SOME CONJECTURES OF GRAFFITI.PC ON THE MAXIMUM ORDER OF INDUCED SUBGRAPHS

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Abstract

The path number of a graph is the maximum number of vertices of an induced path in the graph. One can make analogous definitions for the tree number, forest number, and bipartite number of a graph. We discuss several lower bounds for these invariants that were conjectured by Graffiti.pc, and the relationship between these lower bounds and known theorems or prior conjectures.

Keywords: average distance, bipartite number, decycling number, diameter, forest number, Graffiti.pc, independence number, local independence number, path number, tree number, radius.

Introduction and Key Definitions

Graffiti.pc, a computer program that makes graph theoretical conjectures, was written by E. DeLaViña. This program utilizes conjecture making strategies similar to those found in Graffiti, a program written by S. Fajtlowicz that dates from the mid-1980's. The operation of Graffiti.pc and its similarities to Graffiti are described in [11] and [12]. Some conjectures of Graffiti will be referred to in this paper. A numbered, annotated listing of several hundred of Graffiti's conjectures can be found in [15]. Graffiti has correctly conjectured a number of new bounds for several well-studied graph invariants; bibliographical information on resulting papers can be found in [13].

We limit our discussion to graphs that are simple and finite of order n and size e. The *path number* of a graph G, denoted by $p = p(G)$, is the maximum order of an induced path in the graph. One can make analogous definitions for the *tree number*, *forest number*, and *bipartite number* of a graph G. These invariants are denoted by $t = t(G)$, $f = f(G)$, and $b = b(G)$, respectively.

The purpose of this paper is to state and in some cases prove lower bounds for this suite of related invariants that were recently conjectured by Graffiti.pc. A few of Graffiti.pc's conjectures are "rediscoveries" of prior results, which we include for historical context and motivation, but most are new so far as the authors can discern. In fact, Graffiti.pc has conjectured many more new bounds for these invariants (both upper and lower) than we are able to include in this paper; a numbered, annotated listing of all current conjectures can be found in [9].

Some Other Definitions

We will use the standard notation to denote the minimum and maximum degrees of a graph G ($\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively), and we may sometimes identify G with its set of vertices. We let $\alpha = \alpha(G)$ denote the *independence number* of G. Suppose v is a vertex of G. Then the *local independence number* at v is the independence number of the subgraph induced by the neighbors of v. We use $\mu = \mu(G)$ for the maximum of the local independence numbers taken over all vertices of G.

The following definitions pertain to connected graphs. The *distance between two vertices* of a graph is the length of the shortest path connecting the two vertices. The *average distance* of a graph G, written $\overline{D} = \overline{D}(G)$, is the average distance between pairs of distinct vertices of G. The *eccentricity of a vertex* v is the maximum of the distances from the vertex v to all other vertices of the graph. The *diameter* of a graph G, denoted $d = d(G)$, is the maximum eccentricity of all vertices of the graph. The set of vertices with maximum eccentricity is called the *boundary* of G. The *radius* of a graph, written $r = r(G)$, is the minimum eccentricity of all vertices. The set of vertices with minimum eccentricity is called the *center* of G.

More specialized definitions will be introduced immediately prior to their first appearance in a conjecture. Standard graph-theoretical terms not defined in this paper can be found in [22]. The following proposition summarizes the obvious relationships between our key invariants, as well as α and d; we will refer to it often.

Proposition 1. *Let G be a connected graph. Then*

$$2\alpha \geq b \geq f \geq t \geq p \geq d + 1$$

Motivation, Conjectures, and Main Results

Part i) The path number and the tree number

Many classical results about the tree number and the path number of a graph can be found in a 1986 paper by P. Erdös, M. Saks, and V. Sós [14]. For instance, the following theorem provides a lower bound on the tree number in terms of the order and the size.

Theorem 1. *(Erdös, Saks, and Sós) Let G be a graph. Then*

$$t \geq \frac{2n}{e - n + 3}$$

Moreover, the following result about the path number, sometimes referred to as the Induced Path Theorem [16], appears in the same paper, but the proof given is credited to F. Chung. Graffiti.pc also made this conjecture.

Theorem 2. *(Chung) Let G be a connected graph. Then*

$$p \geq 2r - 1$$

There appears to be little follow-up work in the literature that provides new or improved lower bounds on either the tree number or the path number. However, there do exist at least two different generalizations (but not improvements) of Theorem 2, provided independently by S. Fajtlowicz in [16, Theorem 2], and G. Bacsó and Z. Tuza in [3, Theorem 1]. Another of Bacsó's and Tuza's theorems moreover characterizes the case of equality for Theorem 2. For an integer k, a vertex v of a graph G is called a k-center of G provided each vertex of G is within distance k of v. Thus, the center of G is the set of all r-centers.

Theorem 3. *(Bacsó and Tuza) Let G be a connected graph. Then* $p(G) = 2r(G) - 1$ *if and only if for every connected induced subgraph H of G,* $r(H) \leq r(G)$ *and each vertex of H is within distance* $r(G) - 2$ *of an* $r(G)$*-center of H.*

Fajtlowicz's interest in Theorem 2 was motivated by studying the case of equality for the following theorem, which is an immediate consequence of Theorem 2. The statement of this theorem is the first announced conjecture of Graffiti; two different proofs independent of Theorem 4 are given in [17] and [18].

Theorem 4. *(Graffiti 0, Fajtlowicz and Waller) Let G be a connected graph. Then*

$$\alpha \geq r$$

The characterization of equality for Theorem 4 remains open and is likely difficult; recently, Fajtlowicz proved the following conjecture, which is at least as strong as Theorem 4 (outside of cliques) and sheds some light on its case of equality.

Theorem 5. *(Fajtlowicz [11]) Let G be a connected graph. Then*

$$\alpha \geq r + \mu - 2$$

Graffiti.pc has offered the following opinion on the case of equality for Theorem 2. For a connected graph G, let $dp = dp(G)$ denote the number of pairs of vertices at distance d from one another (i.e. the number of pairs of vertices at diametrical distance).

Conjecture 1. *(Graffiti.pc 36) Let G be a connected graph. Then*

$$p \geq 2r/dp$$

Hence, according to Graffiti.pc's conjecture, one can only have $p = 2r - 1$ if the number of pairs of vertices at diametrical distance is more than one. Clearly, beyond the truth of the conjecture, it would be interesting to discover its relationship to Theorem 3, if any exists.

One of Graffiti.pc's conjectures about the tree number echoes Conjecture 1.

Conjecture 2. *(Graffiti.pc 84) Let G be a connected graph. Then*

$$t \geq 2r/\delta$$

While we do not address the problem of upper bounds in this paper, we did encounter the following upper bound on p in our research, which interested us because it can be stated in terms of invariants familiar to Graffiti.pc.

Theorem 6. *(J. Harant et al. [20]) Let G be a graph. Then*

$$p \leq \frac{4\mu n e}{\delta(2e+n)}$$

Finally, let $dom = dom(G)$ denote the *domination number* of a graph G, i.e. the size of the smallest subset of vertices A such that each vertex of G is either contained in A or adjacent to some vertex of A. Graffiti.pc conjectured the following, which makes an entertaining exercise and which the authors have not seen stated elsewhere. We defer all proofs to the last section.

Proposition 2. *(Graffiti.pc 105) Let G be a non-degenerate connected graph. Then*

$$t \geq \frac{\alpha+1}{dom}$$

Part ii) The forest number

On the other hand, there is a fairly large literature of papers dealing with the forest number of a graph, dating from the 1980's through the present. See, for example, [1], [7], and [23]. Many of these papers are concerned with establishing lower bounds for f in certain restricted classes of graphs, such as cubic, triangle-free graphs or bipartite graphs. Contemporary interest in this invariant has remained significant, in part due to interest in its complementary invariant, known as the *decycling number* (see [4], [5]). The decycling number of a graph is the minimum number of vertices that must be deleted from the graph in order to yield an acyclic graph. Again, the goal of many researchers has been to exactly determine or establish upper bounds on the decycling number in restricted classes of graphs. Some interesting recent results on the forest number are due to N. Alon, D. Mubayi, and R. Thomas and are contained in [1].

Theorem 7. *(Alon, Mubayi, and Thomas) Let G be a graph. Then*

$$f \geq \alpha + \frac{n-\alpha}{(\Delta+1)^2}$$

Another noteworthy result is due to M . Zheng and X. Lu [23].

Theorem 8. *(Zheng and Lu) Let G be a connected, cubic, triangle-free graph with* $n \geq 8$. *Then*

$$f \geq n - \left\lceil \frac{n}{3} \right\rceil$$

Several of Graffiti.pc's conjectures about the forest number of a graph can be construed as attempts to improve Proposition 1. Let $f_1 = f_1(G)$ denote the number of vertices of degree 1 in a graph G. (The authors refer to this invariant as the *frequency of degree one.*) Here is an example of such a conjecture that is not difficult to prove.

Proposition 3. *(Graffiti.pc 47) Let G be a connected graph. Then*

$$f \geq d + f_1 - 1$$

Thus for graphs with frequency of degree one greater than one, Proposition 3 duplicates or improves Proposition 1.

The proof of Proposition 3 is easily extended to prove the following two similar propositions. Let $g = g(G)$ denote the *girth* of a graph G, that is, the minimum order of a cycle in G. By the same token, let $c = c(G)$ denote the *circumference* of G, that is, the maximum order of an induced cycle in G.

Proposition 4. *(Graffiti.pc 48) Let G be a connected graph. Then*

$$f \geq g + f_1 - 1$$

Proposition 5. *Let G be a connected graph. Then*

$$f \geq c + f_1 - 1$$

On the one hand, since diameter is defined for every connected graph, Proposition 3 may be considered better than Proposition 4. On the other, however, there are many graphs for which girth is greater than diameter; some examples include cycles, complete graphs and the Petersen graph. While Proposition 5 is clearly an improvement over Proposition 4, for reasons that remain unclear to the authors it was not conjectured by Graffiti.pc[1].

The following theorem is one of the main results of this paper.

Theorem 9. *(Graffiti.pc 67) Let G be a connected graph. Then*

$$f \geq d + \mu - 2$$

Here again, one may wonder about replacing d with g or c. But this time, for instance, cycles on at least 4 vertices are counterexamples to both $f \geq g + \mu - 2$ and $f \geq c + \mu - 2$.

Both Graffiti.pc and Graffiti are fond of making conjectures about average distance invariants. For example, probably the best known conjecture of Graffiti is the basis for the following theorem due to Chung [8].

Theorem 10. *(Graffiti 2, Chung) Let G be a connected graph. Then*

$$\alpha \geq \overline{D}$$

Let A and B be two subsets of the vertices (not necessarily disjoint) of a connected graph G. Then we use $\overline{D}(A, B)$ to denote the average distance between pairs of distinct vertices, one of which is in A and one of which is in B. Hence $\overline{D}(G, G) = \overline{D}(G) = \overline{D}$.

Conjecture 3. *(Graffiti.pc 74) Let G be a connected graph with boundary B. Then*

$$f \geq 2\overline{D}(B, G)$$

It is easy to naively conclude that this conjecture provides a generalization of Theorem 10, given that $2\alpha \geq f$ as implied by Proposition 1, by assuming that $\overline{D}(B, G) \geq \overline{D}$. Indeed, Graffiti.pc at one time made this very conjecture.

Conjecture 4. *(Graffiti.pc 22) Let G be a connected graph. Then*

$$\overline{D}(B, G) \geq \overline{D}$$

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[1] One explanation may be that what Graffiti.pc actually conjectures is a system of inequalities as described in [12]. For forest number the system of eleven inequalities is listed in [9].

However, this conjecture is false, as demonstrated by the following counterexample, where $\overline{D}(B, G) \approx 2.57$ and $\overline{D} \approx 2.62$.

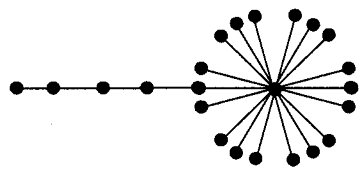

Figure 1: Counterexample to Conjecture 4.

Part iii) The bipartite number

While there are many papers in the literature that explore bounds for maximum induced bipartite subgraphs of a graph, virtually all of these define the maximum subgraph in terms of the number of edges (see, for example, [2], [6], and [19]). One of the few results that pertains to the maximum order of an induced bipartite subgraph is given by Fajtlowicz in [16], as a modest improvement on the lower bound for b implied by the Theorem 2. Once again, Graffiti.pc repeated this conjecture.

Theorem 11. *(Fajtlowicz) Let G be a connected graph. Then*

$$b \geq 2r$$

Fajtlowicz states this theorem as an immediate consequence of his generalization of Theorem 2 mentioned earlier, however, the theorem can be derived directly from Theorem 2, as we shall demonstrate. The theorem can also be derived easily from a result due to O. Favaron, M. Maheo, and J-F. Sacle [18] that we quote in the proof of Theorem 13.

The primary motivation for the authors to include the invariants b, f, t, and p in Graffiti.pc was their curiosity about Theorem 2 along with the following conjecture, which is one of the best known open conjectures of Graffiti. This conjecture was first circulated in 1992 [15], and later by D. West in 1996 [21].

Conjecture 5. *(Graffiti 747) Let G be a connected graph. Then*

$$b \geq 2\overline{D}$$

What makes this conjecture interesting is that unlike Conjecture 3, this conjecture does indeed provide a generalization of Theorem 10, given that $2\alpha \geq b$ as implied by Proposition 1.

The following theorem is a slight strengthening of Theorem 9 for b and is also one of the main results of this paper.

Theorem 12. *(Graffiti.pc 13) Let G be connected graph. Then*

$$b \geq d + \mu - 1$$

The next conjecture is at least as strong as Theorem 11 for graphs other than cliques.

Conjecture 6. *(Graffiti.pc 16) Let G be a connected graph. Then*

$$b \geq 2(r-1) + \mu$$

Conjecture 6 remains open, but we prove the following weaker result.

Theorem 13. *Let G be a connected graph. Then*

$$b \geq 2r + \mu - 5$$

To close, we state the following unnumbered[2] conjecture of Gaffiti.pc. Note that this conjecture, when combined with the fact that $\alpha \geq r + \mu - 2$ from Theorem 5, implies a slightly weaker version of Conjecture 6, namely, $b \geq 2r + \mu - 3$.

Conjecture 7. *(Graffiti.pc) Let G be a connected graph. Then*

$$b \geq r + \alpha - 1$$

Recall by Theorem 4 that $\alpha \geq r$, thus one might interpret Conjecture 7 as an attempt to match or improve Theorem 11 (outside of the case of equality for Theorem 4). This conjecture remains open as well, however, notice that if we apply the conjecture to trees we get the inequality $n \geq r + \alpha - 1$. A stronger version of this inequality is a conjecture of Graffiti and is true for connected graphs in general; it can be proven using the results of [16].

Proposition 6. *(Graffiti 120, Fajtlowicz) Let G be a connected graph. Then*

$$n \geq r + \alpha$$

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[2] Graffitit.pc periodically displays its "preliminary" list of conjectures dynamically while executing. This unnumbered conjecture was eventually removed from the final list of conjectures.

Proofs

Proof of Proposition 2. Let I be a maximum independent set and let D be a minimum dominating set. Clearly $\alpha \geq dom$. Define a function $f : I \rightarrow D$ as follows. If $x \in I \cap D$, then $f(x) = x$. Otherwise, if $x \in I - D$, choose $y \in D$ such that y is adjacent to x. Let $f(x) = y$. Next choose $z \in D$ such that $|f^{-1}(z)|$ is maximum. Now $|f^{-1}(z)| \geq \lceil \frac{\alpha}{dom} \rceil \geq 1$. If $z \in I$, then $|f^{-1}(z)| = 1$, which implies $\frac{\alpha}{dom} = 1$ and $\frac{\alpha + 1}{dom} \leq 2$. Since G is non-degenerate and connected, z is adjacent to some vertex a. Thus $t \geq 2 \geq \frac{\alpha+1}{dom}$. If $z \in D - I$, then $z \notin f^{-1}(z)$ and $f^{-1}(z) \cup \{z\}$ induces a star in G. Moreover,

$$t \geq |f^{-1}(z) \cup \{z\}| = |f^{-1}(z)| + 1 \geq \lceil \frac{\alpha}{dom} \rceil + 1 \geq \frac{\alpha}{dom} + 1 \geq \frac{\alpha + 1}{dom},$$

and we are finished. □

Proof of Propositions 3, 4, and 5. Let D be the set of vertices of a diametric path of G. Then $|D| = d + 1$. Let F be the set of vertices of degree one of the graph. Since the sets D and F have at most two vertices in common and clearly the vertices of $D \cup F$ induce a forest, $f \geq (d + 1) + (f_1 - 2) = d + f_1 - 1$. If the graph is not acyclic, the argument is similar for $f \geq c + f_1 - 1$, from which $f \geq g + f_1 - 1$ follows by transitivity. □

Alternative proof of Theorem 11. By way of contradiction, suppose G is a counterexample. We know by Theorem 2 that there exists an induced path of order at least $2r - 1$, call it P. Now P must have order exactly $b = 2r - 1$, or we are finished. Let c be the unique center vertex of P. Properly color the vertices of P red and green. So the endpoints of P have the same color. But each vertex v of G outside of P must be adjacent to both a red and green vertex of P, or $b \geq 2r$ and G is not a counterexample. Thus v must be adjacent to an interior vertex of P. Then the eccentricity of c in G is $(b - 1)/2 = (2r - 2)/2 = r - 1$, a contradiction. □

Proof of Theorem 13. We first state a lemma due to Favaron, Maheo, and Sacle [18].

Lemma. Let G be a connected graph and suppose c is a vertex in the center of G such that the number of vertices at distance r from c is minimized over the vertices in the center. Then there exist vertices u_i, v_i for $0 < i < r$ such that u_i and v_i are non-adjacent, and the distance from c to both u_i and v_i is i.

Now choose a vertex c in the center of G as described in the lemma. In addition to the vertices u_i, v_i given by the lemma, let $u_0 = c$ and let u_r be any vertex at distance r from c. For even values of i, where $0 \leq i \leq r$, color the vertices u_i, v_i red. For odd values of i, color these vertices green. Because the vertices u_i, v_i are non-adjacent and are both at distance i from c, then no two vertices of the same color can be adjacent to one another. Hence the subgraph B induced by the u_i, v_i is bipartite. It immediately follows that $b \geq |B| \geq 2r$.

Next let v be a vertex such that its local independence is maximum among all vertices of G. Moreover, let N be a largest independent set induced by the neighbors of the v. Assume the distance from v to c is k where k is odd (the argument is symmetrical if k is even). Then clearly each vertex of N is at distance either $k-1, k,$ or $k+1$ from c. Consider the subgraph B' induced by the vertices $(B - \{u_{k-1}, v_{k-1}, u_k, v_k, u_{k+1}, v_{k+1}\}) \cup N \cup \{v\}$. We color the vertices of $(B' \cap B) - (N \cup \{v\})$ as before, but we color the vertices of N red and the vertex v green. It is easy to see that no two vertices of the same color are adjacent to one another, i.e. B' is bipartite. Therefore, $b \geq |B'|$ $\geq |(B - \{u_{k-1}, v_{k-1}, u_k, v_k, u_{k+1}, v_{k+1}\}) \cup N \cup \{v\}| \geq 2r - 6 + \mu + 1$ $= 2r + \mu - 5$, and we are finished. □

Proof of Theorems 9 and 12. Let $D = \{v_1, v_2, \ldots, v_{d+1}\}$ be the set of vertices of a diametric path of G such that v_i is adjacent to v_{i+1}, for $i = 1, 2 \ldots, d$. If the diameter of the graph is one, then the graph is complete and $b = f = 2 > 1 = d + \mu - 1$. Thus we will assume that $d \geq 2$, that is $|D| \geq 3$.

Let v be a vertex such that its local independence is maximum among all vertices of G; we will refer to v as a μ-vertex. Let $N = \{n_1, n_2, \ldots, n_\mu\}$ be a largest independent set induced by the neighbors of the μ-vertex, v. We proceed in two cases. First, we assume that v is a vertex in D, and second that it is not a vertex in D. In either case, since the vertices of D induce a diametric path, we make the following observations.

(1) If a vertex of a set $V - D$ is adjacent to two vertices of D, then as labeled in D, their indices will differ by at most two.

(2) The sets D and N have at most two vertices in common.

Case 1. Suppose that the μ-vertex, v, is a vertex on the diametric path induced by the vertices of D, say $v = v_k$ (see Figure 2).

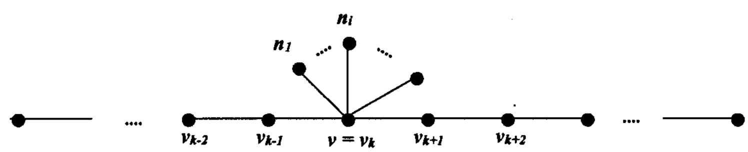

Figure 2: Case 1. Assume v is in D.

If the μ-vertex is an endpoint of the diametric path, without loss of generality, say it is v_1. In this case, each vertex of N must be adjacent to either vertex v_2 or v_3, otherwise the vertices of D do not induce a diametric path. If $v_2 \in N$, the $d+\mu-1$ vertices of $N \cup (D - \{v_1\})$ clearly induce a tree. If $v_2 \notin N$, the $d+\mu-1$ vertices of $N \cup (D - \{v_1, v_2\})$ clearly induce a forest. Thus, for the remainder of this proof we assume that $2 \leq k \leq d$ (that is, the μ-vertex, v, is not an endpoint of a diametric path).

Since we have assumed that the μ-vertex is a vertex v_k of D, by (1) we easily deduce the following.

(3) If any vertex of N is adjacent to vertices of $D - \{v_k\}$, then the only possibilities are among the vertices $v_{k-2}, v_{k-1}, v_{k+1}$, or v_{k+2} (if they exist).

By (2) the vertex sets D and N have at most two vertices in common; thus, we consider three subcases.

Case 1.a. Assume that D and N have exactly two vertices in common. In this case, since the μ-vertex is vertex v_k, clearly the two common vertices must be vertices v_{k-1} and v_{k+1}. If no vertex of $N - D$ is adjacent to a vertex of D, then clearly the vertices of $N \cup D$ induce a tree, in which case $b \geq f \geq t \geq d + \mu - 1$, that is both results follow.

On the other hand, if some vertex n_i of $N - D$ is adjacent to a vertex of D (other than v_k), then by (3) and since v_{k-1} and v_{k+1} are in N, vertex n_i is adjacent to either v_{k-2} or v_{k+2}. We demonstrate a 2-coloration of the vertices of $N \cup D$ from which the result for b follows. Color the vertices of N green (see Figure 3; the boxes will represent green) and color the vertices v_{k-2}, v_k, and v_{k+2} red (see Figure 3; the circles will represent red). By (3), the remaining vertices of D, not yet colored, are easily colored red or green. In this case, all vertices of D and N are colored, and since they have two vertices in common, $b \geq d + \mu - 1$.

Since vertices of $N - D$ are adjacent to at most one of v_{k-2} or v_{k+2} (otherwise the vertices of D do not determine a diametric path), we see that the vertices $N \cup (D - \{v_k\})$ induce a forest. Thus, $f \geq d + \mu - 2$.

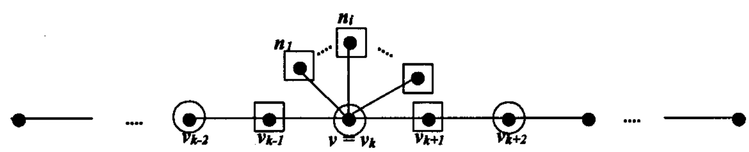

Figure 3: A 2-coloration for a subcase of Case 1.a.

Case 1.b. Assume that D and N have exactly one vertex in common; without loss of generality, suppose that it is vertex v_{k+1}. In this case, since v_{k-1} is not in N, there exists a vertex, say n_i, in N adjacent to v_{k-1} (see Figure 4). By (3) and since v_{k+1} is in N, if a vertex of the set N is adjacent to vertices of $D-\{v_k\}$, then the neighbors are among the vertices v_{k-2}, v_{k-1}, or v_{k+2}. This and (1) allow us to make the following observation.

(4) If a vertex of N is adjacent to two vertices of $D-\{v_k\}$, then the neighbors are the vertices v_{k-2} and v_{k-1}.

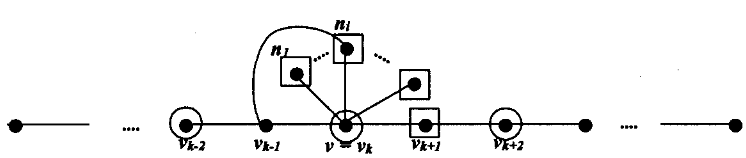

Figure 4: Case 1.b. Sets N and D have only v_{k+1} in common.

Next, we demonstrate a 2-coloration of the $d+\mu-1$ vertices of the set $N \cup (D-\{v_{k-1}\})$. Color the vertices of N (which includes vertex v_{k+1}) green, and color the vertices v_{k-2}, v_k, and v_{k+2} red (see Figure 4). By (3), the remaining vertices on the diametric path, $D-\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$, are easily colored red or green. Since all but one of the vertices of D have been colored, all vertices of N have been colored, and D and N have exactly one vertex in common, $b \geq d+\mu-1$. Moreover, by (4), we see that the vertices $N \cup (D-\{v_{k-1}, v_k\})$ induce a forest. Thus, $f \geq d+\mu-2$.

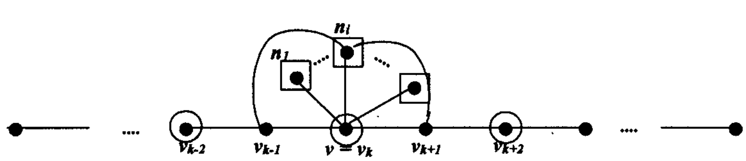

Figure 5: Case 1.c. Sets N and D have no common vertices.

Case 1.c. Suppose that N and D have no common vertices, that is, suppose neither of v_{k-1} or v_{k+1} is in N. In this case, there exists a vertex (or vertices) of N adjacent to v_{k-1} and v_{k+1} (see Figure 5 for example). We will demonstrate a 2-coloration of the $d + \mu - 1$ vertices of $N \cup (D - \{v_{k-1}, v_{k+1}\})$ (see Figure 5). Color the vertices of N green, and color the vertices v_{k-2}, v_k, and v_{k+2} red. By (3), the remaining vertices on the diametric path in $D - \{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$ are easily colored red or green. All but two of the vertices of D have been colored, and all vertices of N have been colored. Since D and N have no vertex in common, $b \geq d + \mu - 1$.

By (1) and (3), a vertex of N is adjacent to at most one of v_{k-2} or v_{k+2}. Thus we see that the vertices $N \cup (D - \{v_{k-1}, v_k, v_{k+1}\})$ induce a forest. Hence, $f \geq d + \mu - 2$.

Case 2. Assume that the μ-vertex, v, is not a vertex on the diametric path induced by the vertices of the set D. By (2) the vertex sets D and N have at most two vertices in common, thus we consider three subcases.

Case 2.a. Assume that D and N have exactly two vertices in common. By (1), and considering N is an independent set, the indices of the two common vertices, as labeled in D, differ by exactly two. Without loss of generality, let the two vertices common to N and D be denoted as v_k and v_{k+2}.

If no vertex of $N - \{v_k, v_{k+2}\}$ is adjacent to vertices of D, then the vertices of $D \cup N$ clearly induce a forest (a path and isolated vertices) on $d + \mu - 1$ vertices, in which case, both results follow.

Since the vertices of D induce a diametric path and the μ-vertex, v, is assumed to be adjacent to both v_k and v_{k+2}, the following observations hold.

(5) If a vertex of $N - \{v_k, v_{k+2}\}$ is adjacent to a vertex of D, then the only possible neighbors are among the vertices in the set $\{v_{k-1}, v_{k+1}, v_{k+3}\}$.

(6) A vertex of N is not adjacent to both of the vertices v_{k-1} and v_{k+3} of D.

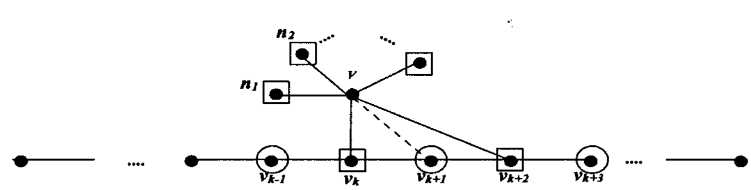

Figure 6: Case 2.a.

In this case we exhibit a 2-coloration of the $d+\mu-1$ vertices of the set $D \cup N$. Color all vertices of N green (indicted by the boxes around vertices as seen in Figure 6), and color vertices v_{k-1}, v_{k+1}, and v_{k+3} red. By (5) the remaining vertices of $D-\{v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ are easily colored red or green. In this case, since we have colored all vertices of N, and all vertices of D, $b \geq d+\mu-1$.

By (5) and (6) a vertex of N is adjacent to at most one vertex of the set $D-\{v_{k+1}\}$. Thus, the vertices $N \cup (D-\{v_{k+1}\})$ induce a forest, from which it follows that $f \geq d+\mu-2$.

Case 2.b. Assume that D and N have exactly one vertex in common. Without loss of generality, let the vertex common to N and D be denoted as v_k. If no vertex of $N-\{v_k\}$ is adjacent to a vertex of D then the vertices of $D \cup N$ clearly induce a forest (a path and isolated vertices) on $d+\mu-1$ vertices, in which case, both results follow.

Since the vertices of D induce a diametric path and the μ-vertex, v, is assumed to be adjacent to v_k, the following observations hold in this case.

(7) If the μ-vertex, v, is adjacent to vertices of D (other than v_k), then the only possible neighbors are among the vertices in the set $\{v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}\}$, but not to all; we indicate these possible edges by dashed lines in Figure 7.

(8) If a vertex of $N-\{v_k\}$ is adjacent to a vertex of D, then the only possible neighbors are among the vertices in the set $\{v_{k-3}, v_{k-2}, v_{k-1}, v_{k+1}, v_{k+2}, v_{k+3}\}$ (these are the vertices labeled in Figure 7).

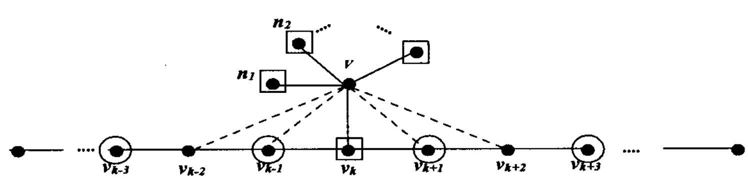

Figure 7: Case 2.b.

By (7) the vertex of D with the smallest index that may be adjacent to vertex v is $k-2$. If v is adjacent to vertex v_{k-2}, then by (1) vertex v is not adjacent to either of the vertices v_{k+1} or v_{k+2}. Further, if v is adjacent to vertex v_{k-2}, then no vertex of N is adjacent to either of the vertices v_{k+2} or v_{k+3} (since the vertices of D induce a diametric path). Thus, in the case that v is adjacent to vertex v_{k-2}, the only vertices of D that we need deliberately demonstrate a 2-coloration for are the vertices $\{v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}\}$ (they are labeled in Figure 8). In this case, we exhibit a 2-coloration of the $d+\mu-1$ vertices of the set

$N \cup \{v\} \cup (D - \{v_{k-2}, v_{k-1}\})$. Color all vertices of N green, and color vertices v, v_{k-3}, and v_{k+1} red. The remaining vertices of the diametric path $D - \{v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but two vertices of D, and we have colored the μ-vertex. Thus $b \geq d + \mu - 1$, and the result follows in this case. In order to show that, in the case that v is adjacent to vertex v_{k-2}, the result for the forest number also holds, we observe that by (1) no vertex of N is adjacent to both vertices v_{k-3} and v_{k+1}; thus, the vertices $N \cup (D - \{v_{k-2}, v_{k-1}\})$ induce a forest, from which $f \geq d + \mu - 2$ follows.

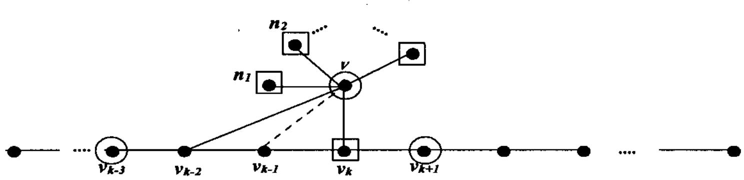

Figure 8: Case 2.b. The subcase that v is adjacent to v_{k-2}.

Next, let us assume that the μ-vertex, v, is not adjacent to vertex v_{k-2} but is instead adjacent to v_{k-1} (as seen in Figure 9). Then by (1), vertex v is not adjacent to vertices v_{k+2} and v_{k+3}; and no vertex of N can be adjacent to vertex v_{k+3} (otherwise the vertices of D do not determine a diametric path). Thus, we need not demonstrate the coloration of vertex v_{k+3}, which by (8) implies that

(9) We need only consider the vertices $\{v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$ of D.

We proceed in this case by considering two further subcases: First that the μ-vertex, v, is adjacent to vertex v_{k+1}; and second that it is not adjacent to vertex v_{k+1}.

Case 2.b.i. Suppose that v is also adjacent to vertex v_{k+1} (as seen in Figure 9). Then the vertices of N cannot be adjacent to vertex v_{k-3} (otherwise D does not determine a diametric path). Thus by (9), the only vertices of D we need focus on are the vertices $\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$ (they are labeled in Figure 9). We exhibit a 2-coloration of the $d + \mu - 1$ vertices of the set $N \cup \{v\} \cup (D - \{v_{k-1}, v_{k+1}\})$. Color all vertices of N green, and color vertices v, v_{k-2}, and v_{k+2} red. The remaining vertices of the diametric path $D - \{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but two vertices of D, and we have colored the vertex v; thus, $b \geq d + \mu - 1$. In order to show that, in this subcase, the result for the forest number also holds, we observe that by (1) no vertex of N is adjacent to both vertices v_{k-2} and v_{k+2}; thus, the vertices $N \cup (D - \{v_{k-1}, v_{k+1}\})$ induce a forest, from which $f \geq d + \mu - 2$ follows.

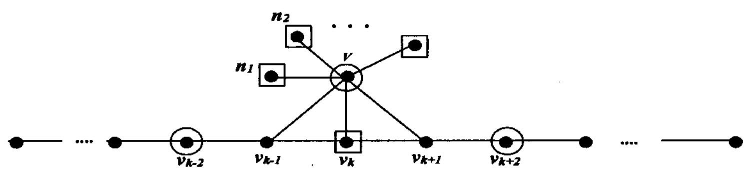

Figure 9: Case 2.b.i. The subcase that v is adjacent to v_{k+1}.

Case 2.b.ii. Suppose that v is not adjacent to vertex v_{k+1}. We are still in the subcase that the μ-vertex, v, is not adjacent to vertex v_{k-2} but is instead adjacent to v_{k-1}. (This arrangement can be seen in Figure 10.)

If a vertex of N is adjacent to vertex v_{k-3}, then no vertex of N is adjacent to vertex v_{k+2} (see Figure 10), and by (1) vertex v cannot be adjacent to v_{k+2}. Thus by (9), among the vertices of D we need only deliberately demonstrate a coloration of the vertices $\{v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}\}$. In this case, we exhibit a 2-coloration of the $d + \mu - 1$ vertices of the set $N \cup \{v\} \cup (D - \{v_{k-2}, v_{k-1}\})$. Color all vertices of N green (illustrated by the boxes around vertices in Figure 10), and color vertices $v, v_{k-3},$ and v_{k+1} red. The remaining vertices of the diametric path $D - \{v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but two vertices of D, and we have colored the vertex v; thus $b \geq d + \mu - 1$. In order to show that, in this subcase, the result for the forest number also holds, we observe that no vertex of N is adjacent to both vertices v_{k-3} and v_{k+1}; thus, the vertices $N \cup (D - \{v_{k-2}, v_{k-1}\})$ induce a forest, from which $f \geq d + \mu - 2$ follows.

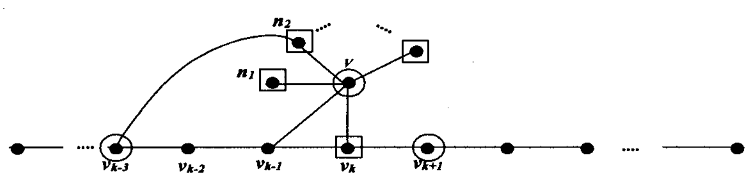

Figure 10: Case 2.b.ii, and a vertex of N adjacent to v_{k-3}.

If no vertex of N is adjacent to vertex v_{k-3} (as seen in Figure 11), then by (9), among the vertices of D we need only deliberately demonstrate a coloration of the vertices $\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$. In this case, we exhibit a 2-coloration of the $d + \mu - 1$ vertices of the set $N \cup \{v\} \cup (D - \{v_{k-1}, v_{k+1}\})$. Color all vertices of N green, and color vertices v, v_{k-2}, and v_{k+2} red. The remaining vertices of the diametric path $D - \{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but two vertices of D, and we have colored the vertex v; thus $b \geq d + \mu - 1$. In order to show that, in this subcase, the result for the forest number also holds, we observe that no vertex of N is adjacent to both vertices v_{k-2} and v_{k+2}; thus, the vertices $N \cup (D - \{v_{k-1}, v_{k+1}\})$ induce a forest, from which $f \geq d + \mu - 2$ follows.

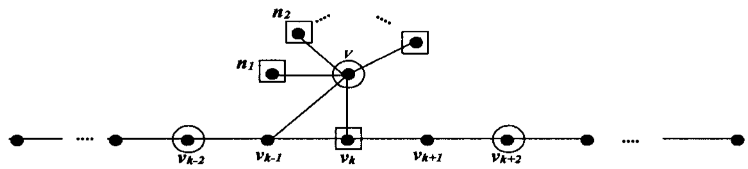

Figure 11: Case 2.b.ii, and no vertex of N adjacent to v_{k-3}.

To complete the proof of case 2.b, we assume that v is not adjacent to either of the vertices v_{k-2} or v_{k-1}. In this case by (7), vertex v can be adjacent only to vertex v_{k+1} or vertex v_{k+2} of $D - \{\, v_k\}$.

If vertex v is adjacent to vertex v_{k+2}, then clearly no vertex of N is adjacent to either vertex v_{k-3} or v_{k-2}, which by observation (8) implies that we need only focus on vertices $\{v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ of the set D. In this case, we exhibit a 2-coloration of the $d + \mu - 1$ vertices of the set $N \cup (D - \{v_{k+2}\})$. Color all vertices of N green, and color vertices v_{k-1}, v_{k+1}, and v_{k+3} red. The remaining vertices of the diametric path $D - \{v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but one vertex of D, and since we are in the case that N and D have exactly one vertex in common, $b \geq d + \mu - 1$. In order to show that, in this subcase, the result for the forest number also holds, we observe that no vertex of N is adjacent to both vertices v_{k-1} and v_{k+3}; thus, the vertices $N \cup (D - \{v_{k+1}, v_{k+2}\})$ induce a forest, from which $f \geq d + \mu - 2$ follows.

If vertex v is not adjacent to vertex v_{k+2} but is instead adjacent to vertex v_{k+1}, then no vertex of N is adjacent to vertex v_{k-3}, which by observation (8) implies that we need only focus on vertices $\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ of D. If some vertex of N is adjacent to vertex v_{k-2}, then no vertex of N can be adjacent to vertex v_{k+3}. In this case we only need focus on vertices $\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$ of D. Similarly if some vertex of N is adjacent to vertex v_{k+3}, then no vertex of N can be adjacent to vertex v_{k-2}. In this case we only need focus on vertices $\{v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ of D. In either case, we easily exhibit a 2-coloration of a $d + \mu - 1$ vertex set; in the former we color $N \cup \{v\} \cup (D - \{v_{k-2},\, v_{k+1}\})$ and in the latter we color $N \cup \{v\} \cup (D - \{v_{k+1},\, v_{k+3}\})$. In order to show that, in these cases, the result for the forest number also holds, we observe that in the former set, no vertex of N is adjacent to both vertices v_{k-2} and v_{k+2}; thus, the vertices $N \cup (D - \{v_{k-1}, v_{k+1}\})$ induce a forest; and for the latter set, we observe that no vertex of N is adjacent to both vertices v_{k-1} and v_{k+3}; thus, the vertices $N \cup (D - \{v_{k+1}, v_{k+2}\})$ induce a forest. Hence, in either case, $f \geq d + \mu - 2$ follows.

Case 2.c. Suppose that the sets N and D have no vertices in common. If the μ-vertex, v, and the vertices of N are not adjacent to vertices of D, then the vertices of $N \cup D \cup \{v\}$ induce a forest (the union of a path and a star), from which is follows that $f \geq d + \mu + 2$. If the μ-vertex, v, is not adjacent to any vertices of D but vertices (or a vertex) of N are adjacent to vertices of D, then we will assume that v_k is the one with the smallest index (as in Figure 12). In case other vertices of N are adjacent to a vertex of D, then the only possible neighbors are among the vertices in the set $\{v_k, v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}\}$ (these are labeled in Figure 12). In this case, we exhibit a 2-coloration of the $d + \mu$ vertices of the set $N \cup \{v\} \cup (D - \{v_{k+1}, v_{k+3}\})$. Color all vertices of N green (illustrated by the boxes around vertices in Figure 12), and color the vertices v_k, v_{k+2}, and v_{k+4} red. The remaining vertices of the diametric path $D - \{v_k, v_{k+1}, v_{k+2}, v_{k+3}, v_{k+4}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but two vertices of D, and we have colored the vertex v; thus $b \geq d + \mu$. Since no vertex of N is adjacent to both vertices v_k and v_{k+4}, the vertices $N \cup (D - \{v_{k+1}, v_{k+2}, v_{k+3}\})$ induce a forest of order $d + \mu - 2$.

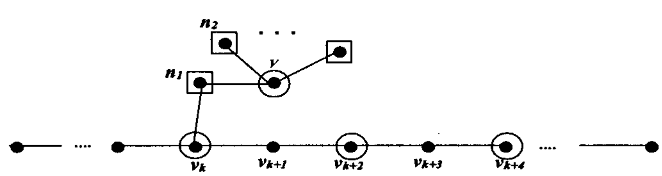

Figure 12: Case 2.c. The subcase v is not adjacent to vertices of D.

In the remainder of Case 2.c, we assume that vertex v has neighbor(s) in D. We will assume that v_k is the one with the smallest index. In this case, since the vertices of D determine a diametric path we make the following observations.

(10) The only possible neighbors of vertex v in D are among the vertices v_k, v_{k+1}, and v_{k+2}.

(11) The only possible neighbors of a vertex of N in D are among the vertices in the set $\{v_{k-3}, v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ (these are labeled in Figure 13).

Furthermore, the assumptions that v_k and v are adjacent, and v_k is not in N, imply the following.

(12) There is some vertex of N that is adjacent to vertex v_k.

Observation (12) and the assumption that the vertices of D determine a diametric path imply the following.

(13) The neighborhood of the vertex set N has at most five elements in D, and if v_j is the one with the smallest index, then the neighbor with the largest index has index at most $j+4$.

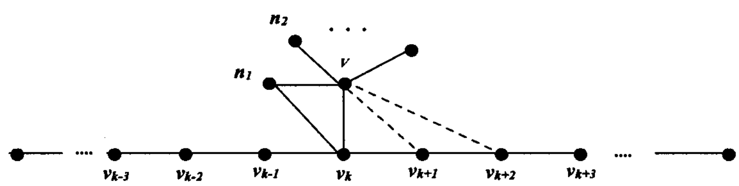

Figure 13: Case 2.c. The subcase v has neighbor(s) in D.

For the rest of this proof we consider the following two subcases: That v is adjacent to only v_k; and that v is adjacent to at least two vertices of $\{v_k, v_{k+1}, v_{k+2}\}$.

Case 2.c.i. Suppose that vertex v is adjacent to vertex v_k but not to vertices v_{k+1} and v_{k+2}. By (12) we know that some vertex of N, say n_1, is adjacent to vertex v_k. Let D^* be the set of vertices of D that are adjacent to vertices of N. By (13) we know that D^* has at most five vertices, and we have assumed that v_k is among them.

If v_k has the smallest index among the vertices of D^*, then by (11) and (13) the vertex with the largest index has index at most $k+3$. Thus in this case, we exhibit a 2-coloration of the $d+\mu$ vertices of the set $N \cup \{v\} \cup (D - \{v_k, v_{k+2}\})$. Color all vertices of N green (illustrated by the boxes around vertices in Figure 14), and color vertices v, v_{k+1}, and v_{k+3} red. The remaining vertices of the diametric path $D - \{v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but two vertices of D, and we have colored the vertex v; thus $b \geq d + \mu$. Moreover, in this case the result for the forest number follows since one can exclude for example v_{k+1}, by which one easily sees that the vertices of $N \cup (D - \{v_k, v_{k+1}, v_{k+2}\})$ induce a forest of order $d + \mu - 2$.

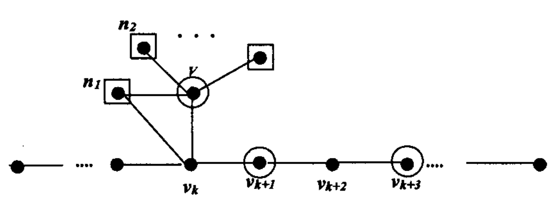

Figure 14: Case 2.c.i.

Next, we consider the case that the vertex of smallest index among the vertices of D^* has index less than k. Let j be the smallest index of the vertices of D^* (where $k-3 \leq j \leq k-1$). Then by (13) the vertex with the largest index has index at most $j+4$. Thus in this case, we exhibit a 2-coloration of the $d+\mu-1$ vertices of the set $N \cup \{v\} \cup (D - \{v_{j+1}, v_{j+2}, v_{j+3}\})$. Color all vertices of N green, and color vertices v_j and v_{j+4} red. The remaining vertices of the diametric path $D - \{v_j, v_{j+1}, v_{j+2}, v_{j+3}, v_{j+4}\}$ are easily colored red or green. In this case we have colored all vertices of N, all but three vertices of D, and we have colored the vertex v; thus $b \geq d+\mu-1$. Moreover, in this case the result for the forest number follows since one can exclude for example v_j, by which one easily sees that the vertices of $N \cup \{v\} \cup (D - \{v_j, v_{j+1}, v_{j+2}, v_{j+3}\})$ induce a forest of order $d+\mu-2$.

Case 2.c.ii. Suppose that vertex v is adjacent to vertex v_k and to at least one of the vertices v_{k+1} or v_{k+2}. By (12) we know that some vertex of N, say n_1, is adjacent to vertex v_k. By (13) we know that we only have to demonstrate a coloration of vertices chosen from at most five consecutive vertices of D, and that v_k is one of these five consecutive vertices. If vertex v is adjacent to vertex v_{k+1}, then the smallest index of the neighbors of N in D is at least $k-2$. If v is adjacent to vertex v_{k+2} then the smallest index of neighbors of N in D is at least $k-1$. Hence, in light of (13) we have that the five consecutive vertices under consideration are among the vertices $\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$. In particular, if there are five vertices of D in the neighborhood of N, they will be either $\{v_{k-1}, v_k, v_{k+1}, v_{k+2}, v_{k+3}\}$; or $\{v_{k-2}, v_{k-1}, v_k, v_{k+1}, v_{k+2}\}$. In either case, we exclude the three interior vertices (that is we do not exclude the smallest and the largest indexed vertices, see Figure 15). Since in either case no single vertex of N is adjacent to both the smallest and largest indexed vertices, then $b \geq f \geq d+\mu-1$.

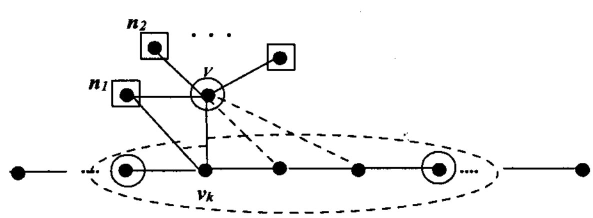

Figure 15: Case 2.c.ii.

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