# Bounds on the $k$-Domination Number of a Graph

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**Abstract**

The $k$-domination number of a graph is the cardinality of a smallest set of vertices such that every vertex not in the set is adjacent to at least $k$ vertices of the set. We prove two bounds on the $k$-domination number of a graph, inspired by two conjectures of the computer program Graffiti.pc. In particular, we show that for any graph with minimum degree at least $2k - 1$, the $k$-domination number is at most the matching number.

## 1. Introduction

For a positive integer $k$, a *$k$-dominating set* of a graph $G$ is a set $S$ of vertices such that every vertex in $V(G) \setminus S$ has at least $k$ neighbors in $S$. For a graph $G$, the minimum cardinality of a $k$-dominating set is called the *$k$-domination number* of $G$, and is denoted $_k(G)$. This invariant was introduced by Fink and Jacobson [6], and has been studied by a number of authors including [2, 4, 5, 7, 8, 9, 10].

We will use some standard terminology from graph theory, for which we refer the reader to [1]. The *independence number* of a graph $G$ is the cardinality of an independent set of maximum size, and will be denoted $(G)$. The *matching number* of a graph $G$ is the cardinality of a matching of maximum size in $G$, and will be denoted $'(G)$.

If $S$ is a set of vertices of $G$, then $G[S]$ will denote the subgraph of $G$ induced by $S$, and $G - S$ will denote the subgraph of $G$ induced by $V(G) \setminus S$. The degree of a vertex $v$ will be denoted $d(v)$, and the minimum degree of $G$ will be denoted $\delta(G)$.

The following result is due to Caro and Roditty [2]:

[1] Work supported in part by the United States Department of Defense and resources of the Extreme Scale Systems Center at Oak Ridge National Laboratory.

[2] Research supported by South African National Research Foundation.

**Theorem 1.** *Let $r$ and $k$ be positive integers. Let $G$ be a graph of order $n$ where $\delta(G) \geq \frac{r+1}{r}k - 1$. Then*

$$_k(G) \leq \frac{r}{r+1}n.$$

We will need the $r = 1$ version of this theorem. Namely:

**Corollary 2.** *Let $G$ be a graph of order $n$ where $\delta(G) \geq 2k - 1$. Then*

$$_k(G) \leq n/2.$$

In this note, we will improve and generalize Corollary 2. Our first result is the following:

**Theorem 3.** *Let $k$ be a positive integer, and $G$ a graph of order $n$. Let $H \subseteq V(G)$ be the set of vertices of degree less than $2k - 1$. Then*

$$_k(G) \leq \ '(G - H) + |H|.$$

If we suppose that $H$ is empty, we get the following succinct result:

**Corollary 4.** *Let $k$ be a positive integer. For any graph $G$ with $\delta(G) \geq 2k - 1$,*

$$_k(G) \leq \ '(G).$$

To see that equality can be achieved in the corollary above, even for graphs that do not have perfect matchings, consider a complete bipartite graph with $2k - 1$ vertices in one part and more than $2k - 1$ vertices in the other part.

Our second result is the following:

**Theorem 5.** *Let $k$ be a positive integer, and $G$ a graph of order $n$. Suppose that in $G$ no two vertices of degree less than $2k - 2$ are adjacent. Let $H \subseteq V(G)$ be the set of vertices of degree less than $2k - 1$. Then*

$$_k(G) \leq \frac{n + \ (G[H])}{2}.$$

Complete graphs of order $2k - 1$ not only demonstrate that this bound is sharp for every $k$, but also provide examples where this bound is sharp while Theorem 3 is very weak and Corollary 2 cannot even be applied. Now, if we suppose $k = 2$ and the graph is bipartite, we get the following result of Fujisawa et al. [7]:

**Corollary 6.** *If $G$ is a bipartite graph, then*

$$_2(G) \leq \tfrac{3}{2} \ (G).$$

These results were inspired by two conjectures of the computer program Graffiti.pc. The program conjectured the special cases of Theorems 3 and 5 where $k = 2$ (see Conjectures 388 and 392a of [3]), and these conjectures were announced at the Southeastern Conference on Combinatorics, Graph Theory and Computing, held in Boca Raton, March 2010.

## 2. Proof of Theorem 3

We need the following folklore result:

**Lemma 7.** *For any graph $G$, $V(G)$ can be partitioned into two parts $S$ and $T$, such that each vertex $v$ in $S$ has at least $d(v)/2$ neighbors in $T$, and each vertex $w \in T$ has at least $d(w)/2$ neighbors in $S$.*

*Proof.* Consider the partition of $V(G)$ into $S$ and $T$ such that the number of edges between $S$ and $T$ is maximized. Then any vertex must have at least half its neighbors in the other part. □

If $G$ is a graph with $\delta(G) \geq 2k-1$ then by Lemma 7 we can partition $V(G)$ into two parts $S$ and $T$ so that each vertex has at least $k$ neighbors in the other part. Corollary 2 is then an easy consequence: both $S$ and $T$ are $k$-dominating sets, and at least one of them has size at most $n/2$.

*Proof of Theorem 3.* By Lemma 7, there exists a partition of the vertices of $G - H$ into two parts $S$ and $T$ such that each vertex in $S$ has at least half its neighbors in $T$, and each vertex in $T$ has at least half its neighbors in $S$.

Let $B$ be the bipartite subgraph of $G - H$ consisting of the edges that are between $S$ and $T$, and let $M$ be a maximum matching in $B$. Let $A$ be the subset of $S$ containing those vertices that are unmatched by $M$. If $A = \emptyset$, then define $C = D = \emptyset$. Otherwise, consider the set of vertices that are reachable from $A$ by an $M$-alternating path. Let $C$ be the subset of $S$ that is reachable in this way, and $D$ the subset of $T$ that is reachable in this way. Note that $A \subseteq C$.

By the maximality of $M$, there is no $M$-augmenting path in $B$ and so all vertices in $D$ are matched by $M$. Furthermore, by the construction, $M$ matches each vertex in $D$ with a vertex in $C$. It follows that

$$|M| = |D \cup (S \setminus C)| .$$

Note that by the construction, there are no edges, in $G - H$, between $C$ and $T \setminus D$. Thus, for any vertex in $C$, at least half its neighbors are in $D$. Similarly, for any vertex in $T \setminus D$, at least half its neighbors are in $S \setminus C$.

Let

$$F = D \cup (S \setminus C) \cup H.$$

We claim that $F$ is a $k$-dominating set for $G$. For, consider any vertex $v$ that is not in $F$. As $v$ is not in $F$, it is not in $H$ either, and so has degree at least $2k-1$. At least half of the neighbors of $v$ are in $F$, since any neighbor of $v$ that belongs to $H$ is in $F$, and at least half of the remaining neighbors are in $F$. It follows that $v$ has at least $k$ neighbors in $F$.

Thus,

$$_k(G) \leq |M| + |H| \leq \ '(G - H) + |H| .$$

□

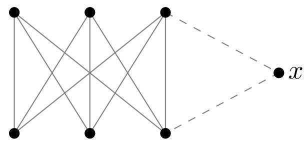


Figure 1: The graph $F$, for $k = 2$.

## 3. Proof of Theorem 5

*Proof of Theorem 5.* By the assumption, the set of vertices of degree less than $2k-2$ is an independent set. Let $I$ be a maximal independent set in $G[H]$ containing all vertices of $G$ of degree less than $2k-2$, and let $J = G - I$. Our strategy will be to construct a $k$-dominating set of $G$ by taking the union of $I$ and a minimum $k$-dominating set of a graph obtained by augmenting $J$ in a certain way so that we may appeal to Corollary 2.

We will use the complete bipartite graph $F = K_{2k-1,2k-1}$ to form a graph $J^*$ with $\delta(J^*) \geq 2k-1$ in the following manner. For each vertex $x$ of $J$ of degree less than $2k-1$: introduce $\lceil ((2k-1) - d_J(x))\,/2 \rceil$ copies of $F$ and attach each to $x$ by two edges such that the ends of the edges are adjacent. See Figure 1.

Let $D^*$ be a minimum $k$-dominating set of $J^*$. We claim that $D^*$ contains at least $2k-1$ vertices from each attached $F$. For, if $D^*$ has less than $k$ vertices from one partite set of the $F$, then it must have every vertex from the other partite set except possibly the vertex $w$ attached to $x$; and if vertex $w \notin D^*$ then at least $k-1$ vertices from the other partite set must be in $D^*$.

Also, we claim that we can choose $D^*$ so that it has exactly $2k-1$ vertices from each attached $F$. For, if it has more, these can be re-arranged to be one partite set and $x$. Further, by considering all the possibilities, it follows that an $x$ attached to a $F$ has exactly one neighbor in that $F$ that is in $D^*$.

Since $\delta(J^*) \geq 2k-1$, we know from Corollary 2 that $|D^*| \leq |V(J^*)|/2$. Set $D = I \cup (D^* \cap V(J))$. From the above it follows that

$$|D| \leq |I| + \frac{|V(J)|}{2} = \frac{n + |I|}{2} \leq \frac{n + \quad (G[H])}{2}. \tag{1}$$

It remains only to show that $D$ is a $k$-dominating set of $G$.

Note that all vertices of $J$ that had no $F$ graphs attached are $k$-dominated by $D$. So, let $x$ be a vertex which had at least one of the $F$ graph attached. If $x \in D$ then there is no problem; so assume $x \notin D$.

By the choice of $I$, $d_G(x) \geq 2k-2$. By the definition of $J$, vertex $x$ has $d_G(x) - d_J(x)$ neighbors in $I$. If $d_G(x) \geq 2k-1$, then $x$ has at most $\lceil (d_G(x) - d_J(x))/2 \rceil$ neighbors in $D^* \setminus D$. Since $x$ is $k$-dominated by $D^*$, it has at least $k - \lceil (d_G(x) - d_J(x))/2 \rceil$ neighbors in $J$, and therefore at least $k$ neighbors in $D$. That is, $x$ is $k$-dominated by $D$.

So assume $d_G(x) = 2k-2$. Then $d_J(x) < 2k-2$, since otherwise we contradict the maximality of $I$. It follows again that vertex $x$ has at least as many neighbors in $I$ as it has in $D^* \setminus D$, and so is $k$-dominated by $D$.

Consequently, $D$ is a $k$-dominating set of $G$, and together with (1), this completes the proof. $\square$

### Acknowledgments

We thank Steve Hedetniemi for helpful discussions.

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