INDEPENDENCE, RADIUS AND PATH COVERINGS IN TREES

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Abstract

We discuss two conjectures of Graffiti generalizing the theorem that the independence number of a simple, connected graph is not less than its radius. The first of these conjectures states that the independence number is not less than the radius plus the path covering number minus one. We demonstrate a family a counterexamples to this conjecture, which we also use to partially resolve one of Graffiti's follow-up conjectures, namely that the independence number is not less than the floor of half the radius plus the path covering number. In particular, we prove a stronger version of this inequality for trees.
Keywords: Graffiti, independence number, path covering number, radius.

Introduction

Graffiti, a computer program that makes conjectures, was written by S. Fajtlowicz. A later version of this program, called Dalmatian, was coauthored with E. DeLaVina. An annotated listing of several hundred of Graffiti's conjectures, dating from the program's inception in the mid-1980's, can be found in [6]. Graffiti has correctly conjectured a number of new bounds for several well-studied graph invariants; bibliographical information on resulting papers can be found in [1].

All graphs considered are simple, connected and finite of order n. We let $\alpha = \alpha(G)$ denote the *independence number* of a graph G. Two simply stated, well known lower bounds for the independence number are expressed in the following two theorems. Theorem 1 results from one of Graffiti's earliest conjectures. Alternative proofs of this theorem are given by O. Favaron (see [6]) and Fajtlowicz (see [4]); the result also follows from a lemma due to F. Chung quoted in [3]. We let $r = r(G)$ denote the *radius* of a graph G.

Theorem 1 *(Fajtlowicz and Waller): Let G be a graph. Then*

$$\alpha \geq r.$$

Theorem 2 is mentioned by L. Lovasz in [8]. A collection of vertex disjoint paths which cover all vertices of a graph G is called a *path covering of the graph* G. The size of a smallest path covering will be called the *path covering number of* G; we use $\rho = \rho(G)$ to denote this number.

Theorem 2 *(Lovasz): Let G be a graph. Then*

$$\alpha \geq \rho.$$

It is easily shown that complete graphs are the only cases of equality for this theorem. In general, both of these invariants are poor approximations for the independence number. For example, consider random graphs with fixed edge probability. Even so, it is interesting to note that these invariants seem complementary to one another in some sense. In particular, the path covering number is a good bound for stars and binary stars, but not for paths, cycles and barbells. On the other hand, the radius is a good bound for paths, cycles and barbells, but not for stars or binary stars.

In light of this observation, the following of Graffiti's conjectures was of particular interest to us.

Conjecture 1 *(Graffiti [5]): Let G be a graph. Then*

$$\alpha \geq r + \rho - 1.$$

The "-1" term is required, for otherwise cliques with more than one vertex would be obvious counterexamples. We will demonstrate counterexamples (though not unique) to this conjecture for all $r \geq 4$. In fact, we will demonstrate a family of trees $\{T_k \mid k = 1, 2, 3, \ldots\}$ where

$$r(T_{2k}) + \rho(T_{2k}) - \alpha(T_{2k}) = k.$$

Note that the inequality

$$\alpha \geq \frac{r}{2} + \frac{\rho}{2}$$

follows immediately from the inequalities $\alpha \geq r$ and $\alpha \geq \rho$. It is thus natural to ask if there exist other pairs of coefficients c_1 and c_2 such that

$$\alpha \geq c_1 \cdot r + c_2 \cdot \rho.$$

Indeed, upon learning of counterexamples to Conjecture 1, Graffiti made the following two conjectures.

Conjecture 2 *(Graffiti): Let G be a graph. Then*

$$\alpha \geq \left\lfloor \frac{r}{2} \right\rfloor + \rho.$$

Conjecture 3 *(Graffiti): Let G be a graph. Then*

$$\alpha \geq r + \frac{\rho - 1}{2}.$$

Figure 1 shows an example of equality for Conjectures 2 and 3. For this graph, it is easy to convince oneself by inspection that $\alpha = 5$, $r = 4$, and $\rho = 3$. Therefore, this graph also serves as a counterexample to Conjecture 1 with $r = 4$.

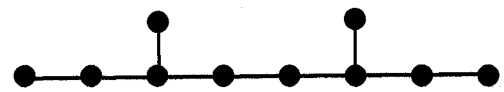

FIGURE 1: Example of equality for Conjectures 2 and 3

Conjectures 2 and 3 remain open in the general case. However, both conjectures are true if restated for trees, as shown by the following two theorems. The proof of Theorem 3 is the main result of this paper. Theorem 4 is proven in [2].

Theorem 3: *Let T be a tree of order more than 2 and suppose $d = d(T)$ is the number of vertices contained on a path in T of maximum length (i.e. d is one more than the diameter of T). Put $x = x(T) = \lfloor d/3 \rfloor$. If x is even, then*

$$\alpha \geq \left(\frac{2x}{3x+2} \right) r + \rho. \tag{1}$$

On the other hand, suppose x is odd. Then

$$\alpha \geq \left(\frac{2x}{3x+1} \right) r + \rho. \tag{2}$$

Moreover, we shall show both bounds are sharp.

Theorem 4 *(DeLaVina, Fajtlowicz and Waller): Let T be a tree. Then*

$$\alpha \geq r + \frac{\rho - 1}{2}.$$

The tree in Figure 1 shows this bound is sharp.

Proofs of Main Results

Counterexamples to Conjecture 1. Consider a path P_{3k} with $3k$ vertices. Enumerate the vertices of P_{3k} from left to right as $v_0, v_1, v_2, \ldots, v_{3k-1}$. Let T_k be the tree on $4k$ vertices formed by attaching a single edge to P_{3k} at each of the vertices v_j where $j = 1 \pmod 3$. Thus T_1 is a star with 3 endpoints; T_2 is formed by taking two copies of T_1 and adding a single edge from an endpoint of one of the stars to an endpoint of the other; and so forth. We will show that for the tree T_{2k},

$$r(T_{2k}) + \rho(T_{2k}) - \alpha(T_{2k}) = k.$$

Proof. Clearly the radius of T_{2k} is $3k$; see Figure 2. For each of the vertices v_j where $j = 1 \pmod 3$ and $0 < j < 6k - 2$, let H_j be the subgraph of T_{2k} induced by the vertices v_j, v_{j+1}, and v_{j+2}, as well as the neighbor s_j of v_j in T_{2k} not contained in P_{6k}. Moreover, let H_0 be the singleton vertex v_0, and let H_{6k-2} be the subgraph induced by the vertices v_{6k-2}, v_{6k-1}, and s_{6k-2}. Suppose I is a maximum independent subset of the vertices of T_{2k}. It is easy to see that $I \cap V(H_j) \leq 2$ for $j \geq 1$. Of course, $I \cap V(H_0) \leq 1$. Since

$$V(G) = V(H_0) \cup V(H_1) \cup V(H_4) \cup \ldots \cup V(H_{6k-2}),$$

it follows that

$$\begin{aligned}
\alpha(T_{2k}) &= |I| \\
&= |(I \cap V(H_0)) \cup (I \cap V(H_1)) \cup (I \cap V(H_4)) \\
&\qquad \cup \ldots \cup (I \cap V(H_{6k-2}))| \\
&\leq 1 + 4k.
\end{aligned}$$

But it is easy to find an independent set of size $4k + 1$ in T_{2k}, so this upper bound is sharp.

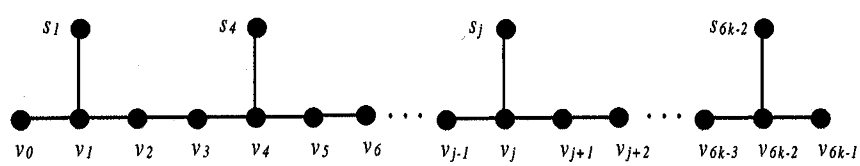

FIGURE 2: T_{2k}, a counterexample to Conjecture 1 when $k \geq 2$

Now if we can show that $\rho(T_{2k}) = 2k + 1$, we are finished. Let $S = \{s_1, s_4, \ldots, s_{6k-2}\}$. Furthermore, let Q be the family of all minimum path coverings of T_{2k}. For every set $C \in Q$, let $s(C)$ be the number of paths in C that are singleton vertices of S. Choose C from Q so that $s(C)$ is maximized.

Claim: $s(C) = |S|$. For suppose otherwise. Then there exists a vertex s_j in S, and a path L in C containing s_j, where the length of L is more than 1. This implies the neighbor v_j of s_j in T_{2k} is also contained in L. Moreover, either v_{j-1} or v_{j+1} is not contained in L. Assume v_{j+1} is not contained in L (the argument is symmetrical in the opposite case). Let M be the path in C containing v_{j+1}. Note that since the degree of v_{j+1} in T_{2k} is at most 2, we have v_{j+1} is an endpoint of M. We now define two new vertex-disjoint paths: the path L' formed by joining $L - s_j$ to M, and $L'' = s_j$. Clearly, $C' = C - \{L, M\} \cup \{L', L''\}$ is contained in Q, because C' covers the vertices of T_{2k} and $|C'| = |C|$. But $s(C') = s(C) + 1$, which contradicts our choice of C.

Our claim implies that each of the paths in C that contains a vertex of S must in fact contain only this vertex. Then $|C| > |S| = 2k$. But since $C \in Q$, there must be only one remaining path in C, namely P_{6k}. Hence $|C| = \rho(T_{2k}) = 2k + 1$. □

Proof of Theorem 3. We start by demonstrating cases of equality for each of the lower bounds (1) and (2) given in the statement of the theorem. Following an argument similar to the one given in the preceding proof, we can see that $\alpha(T_{2k+1}) = 4k + 3$, $r(T_{2k+1}) = 3k + 1$, and $\rho(T_{2k+1}) = 2k + 2$. Next, consider the tree T_k^* formed by adding a single edge to exactly one of the endpoints of the maximum path P_{3k} in T_k. Again following an argument similar to the one given in the preceding proof, we can see that $\alpha(T_{2k+1}^*) = 4k + 3$ and $\rho(T_{2k+1}^*) = 2k + 2$, but $r(T_{2k+1}^*) = 3k + 2$. Therefore, putting

$$x = x(T_{2k+1}^*) = \lfloor d(T_{2k+1}^*)/3 \rfloor = 2k + 1\,,$$

we get

$$\begin{aligned}\alpha(T_{2k+1}^*) &= 4k + 3 \\ &= \left(\frac{2k+1}{3k+2}\right)(3k + 2) + (2k + 2) \\ &= \left(\frac{2x}{3x+1}\right) r(T_{2k+1}^*) + \rho(T_{2k+1}^*).\end{aligned}$$

Now, consider the tree T_k' formed by adding a single edge to each of the endpoints of the maximum path P_{3k} in T_k. For example, the tree in Figure 1 is T_2'. Following an argument similar to the one given in the preceding proof, we can see that $\alpha(T_{2k}') = 4k + 1$ and $\rho(T_{2k}') = 2k + 1$, but $r(T_{2k}') = 3k + 1$. Therefore, again putting

$$x = x(T_{2k}') = \lfloor d(T_{2k}')/3 \rfloor = 2k\,,$$

we get

$$\begin{aligned}\alpha(T_{2k}') &= 4k + 1 \\ &= \left(\frac{2k}{3k+1}\right)(3k + 1) + (2k + 1) \\ &= \left(\frac{2x}{3x+2}\right) r(T_{2k}') + \rho(T_{2k}').\end{aligned}$$

Note that the three families of trees discussed above, T_{2k} ($k \geq 2$), T^*_{2k+1} ($k \geq 1$), and T'_{2k} ($k \geq 1$) provide counterexamples to Conjecture 1 for all $r \geq 4$ and $\rho \geq 3$.

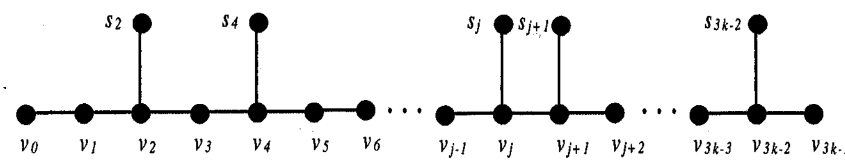

FIGURE 3: An example of a tree R in $\mathcal{R}_k$

Consider a path P_{3k} with $3k$ vertices. Let $\mathcal{R}_k$ be the family of trees formed by attaching at most a single edge to any of the interior vertices of P_{3k}. See Figure 3 for an example of a tree in $\mathcal{R}_k$. Likewise, for $k \geq 1$, let $\mathcal{R}^*_k$ be the family of trees formed by attaching at most a single edge to any of the interior vertices of a path P_{3k+1} with $3k+1$ vertices, and moreover let $\mathcal{R}^{**}_k$ be the family of trees formed by attaching at most a single edge to any of the interior vertices of a path P_{3k+2} with $3k+2$ vertices. We next state two lemmas useful in simplifying the proof of Theorem 3. The proofs of both lemmas are rather technical; we defer them to the last section.

Lemma 1: *Suppose $R \in \mathcal{R}_k$. If k is even, then*

$$\alpha(R) \geq k + \rho(R) = \frac{2}{3}r(R) + \rho(R).$$

On the other hand, if k is odd, then

$$\alpha(R) \geq k + \rho(R) = \left(\frac{2k}{3k-1}\right)r(R) + \rho(R).$$

Lemma 2: *Suppose $R \in \mathcal{R}^*_k \cup \mathcal{R}^{**}_k$. If k is even, then*

$$\alpha(R) \geq \left(\frac{2k}{3k+2}\right)r(R) + \rho(R).$$

On the other hand, if k is odd, then

$$\alpha(R) \geq \left(\frac{2k}{3k+1}\right)r(R) + \rho(R).$$

Conclusion of proof of Theorem 3. By way of contradiction, assume there exist trees of order more than two for which the theorem is not true. Assume T is such a counterexample with the minimum number of vertices. Let $d = d(T)$ and $x = x(T)$ be defined as in the statement of the theorem. Moreover, let P be a path of maximum length in T, and let R be a subtree of T with the maximum number of vertices such that R contains P and $R \in \mathcal{R}_x \cup \mathcal{R}_x^* \cup \mathcal{R}_x^{**}$. Note that $r(R) = r(T)$, and P must have $3x$, $3x+1$, or $3x+2$ vertices. If x is even, then Lemmas 1 and 2 imply that inequality (1) holds for R, i.e.

$$\alpha(R) \geq \left(\frac{2x}{3x+2}\right) r(R) + \rho(R).$$

If x is odd, then Lemmas 1 and 2 imply that inequality (2) holds for R, i.e.

$$\alpha(R) \geq \left(\frac{2x}{3x+1}\right) r(R) + \rho(R).$$

Of course $T \neq R$, otherwise we have a contradiction. Hence, there are two possibilities. Either there exists a vertex s in T that is not in R but which is adjacent to some interior vertex v of P; or, there exists a vertex t in T that is not in R but which is adjacent to some vertex s of R not in P. In the second case, s in turn must be adjacent to some interior vertex v of P. See Figure 4. In either case, suppose the edge joining s and v were deleted from T. Let S' be the connected component of the resulting graph containing s, and let T' be the connected component containing v. Because T' contains P, we have $r(T') = r(R) = r(T)$, and by our choice of T, either inequality (1) or inequality (2) must hold for T'.

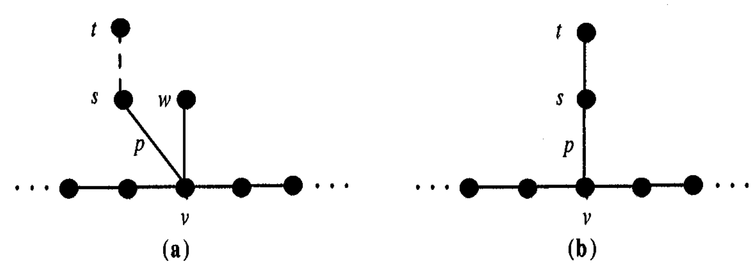

FIGURE 4: Cases when $T \neq R$

Next, suppose $I_{T'}$ is a maximum independent subset of the vertices of T', and $C_{T'}$ is a minimum path covering of T'. We consider three cases.

Case 1: S' is a singleton edge. Let t be the neighbor of s in S', and label the single-edge path joining s and v as p. See Figure 4. Then $I_{T'} \cup \{t\}$ is an independent subset of the vertices of T, and $C_{T'} \cup \{p\}$ is a path covering of T. Since $\alpha(T) \geq |I_{T'} \cup \{t\}| = \alpha(T') + 1$, $r(T') = r(T)$, and

$\rho(T) \leq |\mathcal{C}_{T'} \cup \{p\}| = \rho(T') + 1$, then either inequality (1) or inequality (2) must hold for T, a contradiction.

Case 2: S' is the singleton vertex s. Since R was chosen to contain the maximum number of vertices possible, this implies there exists an endpoint w of T that is adjacent to v, that is not contained in P, but which must be contained in R. For otherwise, s would be contained in R. See Figure 4(a). If we assume $v \notin I_{T'}$, then $I_{T'} \cup \{s\}$ is an independent subset of the vertices of T, and $\mathcal{C}_{T'} \cup \{s\}$ is a path covering of T. However, if $v \in I_{T'}$, then $I_{T'} - \{v\} \cup \{s, w\}$ is an independent subset of the vertices of T, and again $\mathcal{C}_{T'} \cup \{s\}$ is a path covering of T. In any event, like in Case 1, we arrive at a contradiction.

Case 3: S' has order more than two. By our choice of T, either inequality (1) or inequality (2) must hold for S'. Let $I_{S'}$ be a maximum independent subset of the vertices of S', and let $\mathcal{C}_{S'}$ be a minimum path covering of S'. Either of the inequalities (1) or (2), though, imply that $\alpha(S') - 1 \geq \rho(S')$. Now $I_{T'} \cup I_{S'} - \{s\}$ is an independent subset of the vertices of T, and $\mathcal{C}_{T'} \cup \mathcal{C}_{S'}$ is a path covering of T. Since $\alpha(T) \geq |I_{T'} \cup I_{S'} - \{s\}| \geq \alpha(T') + \alpha(S') - 1$, $r(T') = r(T)$, and $\rho(T) \leq |\mathcal{C}_{T'} \cup \mathcal{C}_{S'}| = \rho(T') + \rho(S')$, then either inequality (1) or inequality (2) must hold for T, once more a contradiction. □

Remaining Proofs

Consider a path P_{3k} with $3k$ vertices. Enumerate the vertices of P_{3k} from left to right as $v_0, v_1, v_2, \ldots, v_{3k-1}$. Now suppose $R \in \mathcal{R}_k$, and let v_j be an interior vertex of P_{3k} (i.e. $1 \leq j \leq 3k-2$). Then if v_j has a neighbor not on P_{3k}, we will denote this neighbor vertex by s_j. Further, for any path P of a path covering $\mathcal{C}$ of $R \in \mathcal{R}_k$, we will refer to it as P_{v_i} (or P_{s_i}), where v_i (or s_i) is the endpoint of the path with the smallest index. This endpoint is referred to as the *left endpoint* of the path. If both v_i and s_i are endpoints of a path P in $\mathcal{C}$, we will arbitrarily label P as P_{v_i} or P_{s_i}. See Figure 5 for an example of such a path cover labeling.

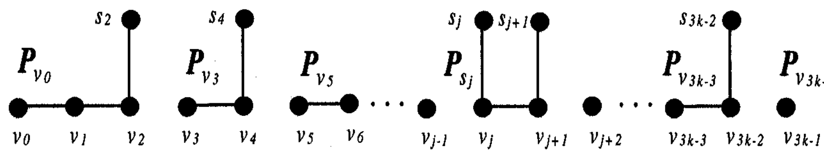

FIGURE 5: Path cover labeling

Lemma 3: *Suppose $R \in \mathcal{R}_k$. For a minimum path covering $\mathcal{C}$ of R, put $L_c = \{v \mid v$ is the left endpoint of some path in $\mathcal{C}\}$. Then there exists a minimum path covering $\mathcal{C}$ of R with the following properties:*

i) *For $0 \le j < 3k - 2$, if $v_j \in V(P_{3k}) \cap L_c$, then $|V(P_{v_j})| \ge 3$.*
ii) *For any $s_j \in L_c$ with $j \ne 3k - 2$, $|V(P_{s_j})| \ge 4$.*

Proof. Let $\mathcal{C}$ be a minimum path covering of R. In finding a minimum path covering with properties *i)* and *ii)*, we will exchange paths in $\mathcal{C}$ as necessary, proceeding left to right. That is, we consider left endpoints of paths in $\mathcal{C}$ by their subscripts in increasing order, thus assuring that one alteration of $\mathcal{C}$ does not "undo" another. Suppose that $v_j \in V(P_{3k}) \cap L_c$ for $0 \le j < 3k - 2$. If P_{v_j} is a singleton vertex, then clearly vertex s_{j+1} must exist and furthermore must be a left endpoint of a path in $\mathcal{C}$, for otherwise $\mathcal{C}$ would not be a minimum path covering. We put $P^-_{v_{j+2}}$ to be the path determined by removing vertices v_{j+1} and s_{j+1} from path $P_{s_{j+1}}$. Note that vertex v_{j+1} must have been on $P_{s_{j+1}}$ by the minimality of $\mathcal{C}$. Next, we put $P^+_{v_j}$ to be the path determined by extending P_{v_j} to include the vertices v_{j+1} and s_{j+1}. In this case, the altered path covering $(\mathcal{C} - \{P_{v_j}, P_{s_{j+1}}\}) \cup \{P^+_{v_j}, P^-_{v_{j+2}}\}$ is also a minimum covering; see Figure 6. However, if it is the case that P_{v_j} is the singleton edge (v_j, v_{j+1}), then again by the minimality of $\mathcal{C}$, s_{j+1} does not exist, but s_{j+2} must exist and moreover must be a left endpoint of a path in $\mathcal{C}$. Similarly, in this case the altered path covering $(\mathcal{C} - \{P_{v_j}, P_{s_{j+2}}\}) \cup \{\{P^+_{v_j}, P^-_{v_{j+3}}\}$ is also a minimum path covering, where $P^-_{v_{j+3}}$ and $P^+_{v_j}$ are respectively determined by removing the vertices v_{j+2} and s_{j+2} from $P_{s_{j+2}}$ and, in turn, appending them to P_{v_j}. Next, if P_{v_j} is the singleton edge (v_j, s_j), then we relabel it P_{s_j} instead and show that property *ii)* holds for P_{s_j}, which we demonstrate below. Thus, part *i)* is established.

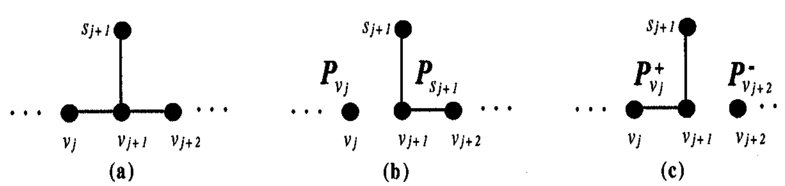

FIGURE 6: Path exchange

For $j \ne 3k - 2$, suppose that $s_j \in L_c$. If P_{s_j} is a singleton vertex, clearly v_j is not a left endpoint of a path of $\mathcal{C}$. Let P_* be the path in $\mathcal{C}$ incident with vertex v_j. Then a new minimum path covering can be constructed by removing v_{j+1} and the portion of path P_* that follows v_{j+1}, and attaching s_j to the end of P_* instead. Let us denote this modification of P_* by P^+_*. The portion of P_* that was detached, call it P_*^-, is now the new path $P_{v_{j+1}}$. The altered path covering $(\mathcal{C} - \{P_{s_j}, P_*\}) \cup \{\{P_*^+, P_*^-\}$ is obviously a minimum path covering. Let us observe that if the left endpoint of P_*^+ is a v_i, then it together with v_j and s_j determine a path on at least three vertices. If, on the other hand, the left endpoint of P_*^+ is an s_i, then it together with v_i, v_j and s_j determine a path on at least four vertices. Next, suppose that P_{s_j} is the singleton edge (s_j, v_j). Then by the minimality of $\mathcal{C}$, clearly v_{j+1} is not a left endpoint of a path in $\mathcal{C}$, and thus s_{j+1} exists. Again, a new minimum path covering can be constructed by removing v_{j+1} and s_{j+1} from the path $P_{s_{j+1}}$, and attaching them to P_{s_j}. Lastly, suppose that P_{s_j} is a path containing three vertices. By our convention for path labelings of a path covering for R, the three vertices are s_j, v_j and v_{j+1}. Then by the minimality of $\mathcal{C}$, clearly s_{j+1}does not exist and v_{j+2} is not a left endpoint of a path in $\mathcal{C}$. Thus s_{j+2} exists. Again, a new minimum path covering can be constructed by removing v_{j+2} and s_{j+2} from path $P_{s_{j+2}}$ and attaching them to P_{s_j}. This completes the proof of the lemma. □

Proof of Lemma 1. Suppose $R \in \mathcal{R}_k$ and let $\mathcal{C}$ be a minimum path covering of R. We put $L_c = \{v \mid v \text{ is the left endpoint of some path in } \mathcal{C}\}$, and assume that $\mathcal{C}$ has the properties *i)* and *ii)* as asserted by Lemma 3. Next, we introduce notation similar to that used in the proof of the counterexamples. That is, for each of the vertices v_j where $j = 1 \pmod 3$ and $1 \leq j \leq 3k-2$, we let H_j be the subgraph of R induced by the vertices v_{j-1}, v_j, and v_{j+1}, as well as any s_i that exist in R for $j-1 \leq i \leq j+1$. See Figure 7 for an example of this notation. We observe that by the definition of $\mathcal{R}_k$, there are k such subgraphs. Further, it is easily verified that $k = 2r(R)/3$ if k is even, and that $k = 2kr(R)/(3k-1)$ whenever k is odd.

We will show that for each of the k subgraphs $H_1, H_4, \ldots, H_{3k-2}$ we can select a vertex $b_j \in V(H_j)$ not adjacent to any vertex of L_c nor to any other selected b_i. The collection of such b_i will be denoted as B. Since L_c is an independent set, and $L_c \cup B$ will be shown to also be an independent set, the lemma will follow.

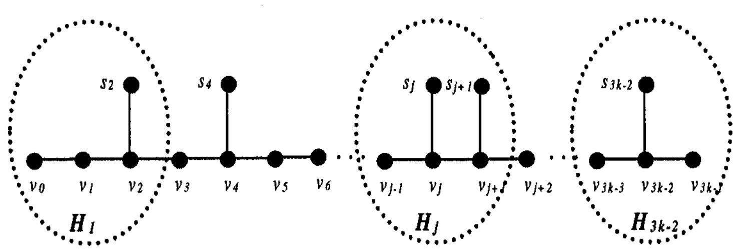

FIGURE 7: The k subgraphs H_1, H_4, … , H_{3k-2}

For each of $H_1, H_4, H_7, \ldots, H_{3k-5}$, we consider five cases. Before we proceed we observe that in each of the following cases we never select vertex v_{i-1} of H_i for the set B. This will insure that B is an independent set.

Case 1: Suppose that $V(H_i) \cap L_c = \emptyset$. Then we put v_i into B, so that L_c and B remain disjoint and v_i is not adjacent to any vertex of L_c.

Case 2: Suppose that $v_{i-1} \in L_c$. If s_i exists, then by properties *i)* and *ii)* of $\mathcal{C}$, the path $P_{v_{i-1}}$ must terminate at vertex s_i. In this case, we put s_i into the set B. Since the only neighbor of s_i is v_i, it follows s_i is not adjacent to any vertex of L_c, and L_c and B remain disjoint. If s_i does not exist but s_{i+1} does, then by properties *i)* and *ii)* of $\mathcal{C}$, the path $P_{v_{i-1}}$ must terminate at vertex s_{i+1}. In this case, we put s_{i+1} into B and clearly B and L_c have the desired properties. Finally, if neither s_i nor s_{i+1} exist, then by property *i)* of $\mathcal{C}$, the path $P_{v_{i-1}}$ contains vertices v_i and v_{i+1}. We can put v_{i+1} into B, since v_{i+2} cannot be a left endpoint of a path in $\mathcal{C}$.

Case 3: Suppose that either $v_i \in L_c$ or $v_{i+1} \in L_c$. By the minimality of $\mathcal{C}$, if $v_i \in L_c$ then s_{i-1} exists. Moreover, by properties *i)* and *ii)* we have $v_{i-1} \notin L_c$ and $s_{i-1} \notin L_c$. We put s_{i-1} into B, and in this case it is clear that B and L_c are as desired. The case $v_{i+1} \in L_c$ is similar.

Case 4: Suppose that either $s_{i-1} \in L_c$ or $s_i \in L_c$. By property *ii)* of $\mathcal{C}$, if $s_{i-1} \in L_c$ then $P_{s_{i-1}}$ passes through s_{i-1}, v_{i-1}, v_i, and either v_{i+1} or s_i (if it exists). If $P_{s_{i-1}}$ terminates at s_i then we put s_i into B, otherwise we put v_i into B. If $s_i \in L_c$ then the argument is similar, except that of course we will put either s_{i+1} or v_{i+1} into B.

Case 5: Suppose that $s_{i+1} \in L_c$. Then by our path labeling convention and property *ii)* of $\mathcal{C}$, $P_{s_{i+1}}$ does not pass through either vertex v_{i-1} nor v_i. Moreover, by property *i)* of $\mathcal{C}$, neither vertex v_{i-1} nor v_i is a left endpoint of a path in $\mathcal{C}$. Thus in this case, we can put v_i into B.

Finally, we consider H_{3k-2}. If either of the vertices s_{3k-3} or v_{3k-3} is a left endpoint, then we can find a vertex in H_{3k-2} for the set B, by duplicating arguments from *Cases 2* or *4*. So let us assume that neither s_{3k-3} or v_{3k-3} is in L_c. If s_{3k-3} exists, then we can put it into B. If s_{3k-3} does not exist, then since v_{3k-3} is not a left endpoint, v_{3k-2} is not a left endpoint either by the minimality of $\mathcal{C}$. The path P_* of $\mathcal{C}$ passing through the vertices v_{3k-3} and v_{3k-2} must terminate at either v_{3k-1} or s_{3k-2} (if it exists). In either case, the terminus of P_* can be put into B. □

Proof of Lemma 2. We consider two main cases for k.

Case I: Assume k is odd.

Case I.1: Assume $R \in \mathcal{R}_k^*$. Let R' be the graph formed by deleting the vertices v_{3k} and s_{3k-1} (if it exists) from R. See Figure 8. Then $R' \in \mathcal{R}_k$. Since k is odd, $r(R') = (3k-1)/2$, $r(R) = (3k+1)/2$, and it is easily verified that

$$\left(\frac{2k}{3k-1}\right)r(R') = \left(\frac{2k}{3k+1}\right)r(R). \qquad (3)$$

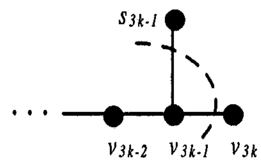

FIGURE 8: If $R \in \mathcal{R}_k^*$, R' in the case s_{3k-1} exists

Case I.1.i: Suppose s_{3k-1} exists. Let $I_{R'}$ be a maximum independent subset of the vertices of R', and let $\mathcal{C}_{R'}$ be a minimum path covering of R'. If $v_{3k-1} \notin I_{R'}$, then $I_{R'} \cup \{s_{3k-1}\}$ is an independent subset of the vertices of R of size $\alpha(R')+1$. If $v_{3k-1} \in I_{R'}$, then $(I_{R'} - \{v_{3k-1}\}) \cup \{v_{3k}, s_{3k-1}\}$ is an independent subset of the vertices of R of size $\alpha(R')+1$. Let L be the path in $\mathcal{C}_{R'}$ that contains v_{3k-1}. Since v_{3k-1} is an endpoint of R', then v_{3k-1} is an endpoint of L. Let L' be the path in R formed by joining the vertex v_{3k} to L, and let $L'' = s_{3k-1}$. Then $(\mathcal{C}_{R'} - \{L\}) \cup \{L', L''\}$ is a path covering of R of size $\rho(R')+1$. Therefore by Lemma 1 and (3),

$$\begin{aligned} \alpha(R) &\geq \alpha(R')+1 \\ &\geq \left(\frac{2k}{3k-1}\right)r(R') + \rho(R') + 1 \\ &\geq \left(\frac{2k}{3k+1}\right)r(R) + \rho(R). \end{aligned}$$

Case I.1.ii: Suppose s_{3k-1} does not exist. Let $I_{R'}$ be a maximum independent subset of the vertices of R', and let $\mathcal{C}_{R'}$ be a minimum path covering of R'. Thus $I_{R'}$ is an independent subset of vertices of R as well. Let L be the path in $\mathcal{C}_{R'}$ that contains v_{3k-1}. Since v_{3k-1} is an endpoint of R', then v_{3k-1} is an endpoint of L. Let L' be the path in R formed by joining the vertex v_{3k} to L. Then $(\mathcal{C}_{R'} - \{L\}) \cup \{L'\}$ is a path covering of R of size $\rho(R')$. Hence, by Lemma 1 and (3) as before,

$$\begin{aligned} \alpha(R) &\geq \alpha(R')+1 \\ &\geq \left(\frac{2k}{3k-1}\right)r(R') + \rho(R') \\ &\geq \left(\frac{2k}{3k+1}\right)r(R) + \rho(R). \end{aligned}$$

Case I.2: Next, assume $R \in \mathcal{R}_k^{**}$. Of course, s_{3k-1} and s_{3k} may or may not exist, but in the following argument, we will assume both these vertices exist. See Figure 9. The argument can be trivially modified if this is not the situation. Let R' be the graph formed by deleting the vertices v_{3k}, v_{3k+1}, s_{3k-1}, and s_{3k} from R; see Figure 9. Then once more $R' \in \mathcal{R}_k$, and since k is odd, $r(R') = (3k-1)/2$, $r(R) = (3k+1)/2$, and it is easily verified that

$$\left(\frac{2k}{3k-1}\right) r(R') = \left(\frac{2k}{3k+1}\right) r(R). \qquad (4)$$

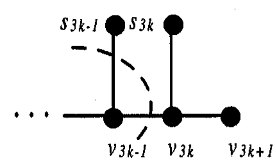

FIGURE 9: If $R \in \mathcal{R}_k^{**}$, R' in the case both s_{3k-1} and s_{3k} exist

Let $I_{R'}$ be a maximum independent subset of the vertices of R', and let $\mathcal{C}_{R'}$ be a minimum path covering of R'. Then $I_{R'} \cup \{v_{3k+1}\}$ is an independent subset of the vertices of R of size $\alpha(R') + 1$. Let L be the path in $\mathcal{C}_{R'}$ that contains v_{3k-1}. Since v_{3k-1} is an endpoint of R', then v_{3k-1} is an endpoint of L. Let L' be the path in R formed by joining the vertex s_{3k-1} to L, and let L'' be the path in R joining the vertices s_{3k}, v_{3k}, and v_{3k+1}. Then $(\mathcal{C}_{R'} - \{L\}) \cup \{L', L''\}$ is a path covering of R of size $\rho(R') + 1$. Therefore, again by Lemma 1 and (4),

$$\begin{aligned} \alpha(R) &\geq \alpha(R') + 1 \\ &\geq \left(\frac{2k}{3k-1}\right) r(R') + \rho(R') + 1 \\ &\geq \left(\frac{2k}{3k+1}\right) r(R) + \rho(R). \end{aligned}$$

Case II: Now we consider the case where k is even.

Case II.1: If we assume $R \in \mathcal{R}_k^{*}$, we can follow an argument similar to the odd *Case I.1* to show

$$\alpha(R) \geq \frac{2}{3} r(R) + \rho(R).$$

Case II.2: Likewise, if we assume $R \in \mathcal{R}_k^{**}$ and either s_{3k} exists, or neither s_{3k-1} nor s_{3k} exist, it is straightforward to conclude the proof. So let us assume the remaining possibility:

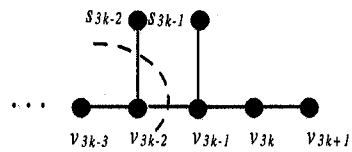

FIGURE 10: If $R \in \mathcal{R}_k^{**}$, R' in the case both s_{3k-1} and s_{3k} exist

Case II.3: $R \in \mathcal{R}_k^{**}$ and s_{3k-1} exists, but s_{3k} does not exist. Let R'' be the graph formed by deleting the vertices v_{3k-1}, v_{3k}, v_{3k+1}, s_{3k-1}, and s_{3k-2} (if it exists) from R. See Figure 10. Then $R'' \in \mathcal{R}_{k-1}^{**}$. It is easy to see that $\rho(R) \leq \rho(R'') + 1$, $\alpha(R) \geq \alpha(R'') + 2$, and $r(R) = r(R'') + 2$. But by Lemma 1 applied to R'' we have

$$\begin{aligned}
\alpha(R) &\geq \alpha(R'') + 2 \\
&\geq \left(\frac{2k-2}{3k-2}\right) r(R'') + \rho(R'') + 2 \\
&= \left(\frac{2k-2}{3k-2}\right)\left(\frac{3k-2}{2}\right) + 1 + \rho(R'') + 1 \\
&= \left(\frac{2k}{3k+2}\right)\left(\frac{3k+2}{2}\right) + \rho(R'') + 1 \\
&\geq \left(\frac{2k}{3k+2}\right) r(R) + \rho(R),
\end{aligned}$$

and we are finished. □

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