A NOTE ON DOMINATING SETS AND AVERAGE DISTANCE

ERMELINDA DELAVIÑA, RYAN PEPPER, AND BILL WALLER
UNIVERSITY OF HOUSTON – DOWNTOWN, HOUSTON, TX, 77002

Abstract. We show that the total domination number of a simple connected graph is greater than the average distance of the graph minus one-half, and that this inequality is best possible. In addition, we show that the domination number of the graph is greater than two-thirds of the average distance minus one-third, and that this inequality is best possible. Although the latter inequality is a corollary to a result of P. Dankelmann, we present a short and direct proof.

1. Introduction and Key Definitions

Let $G = (V, E)$ be a simple connected graph of finite of order $|V| = n$. Although we may identify a graph G with its set of vertices, in cases where we need to be explicit we write $V(G)$ to denote the vertex set of G. A set D of vertices of a graph G is called a *dominating set* provided each vertex of $V - D$ is adjacent to a member of D. The *domination number* of G, denoted $\ = \ (G)$, is the cardinality of a smallest dominating set in G. Likewise, a set D of vertices is called a *total dominating set* provided each vertex of V is adjacent to a member of D. The *total domination number* of G, denoted ${}_t = {}_t(G)$, is the cardinality of a smallest total dominating set in G. The *distance* between two vertices u and v in G is the length of a shortest path in G connecting u and v. The *Wiener index* or *total distance* of G, denoted by $W = W(G)$, is the sum of all distances between unordered pairs of distinct vertices of G [5]. The *average distance* of G, denoted by $\bar{D} = \bar{D}(G)$, is $2W/[n(n-1)]$. Put another way, this number gives, on average, the distance between a pair of vertices of G. Unless stated otherwise, when we refer to a subgraph of G, we mean an induced subgraph.

The total domination number of a graph was first introduced in [2]. This invariant remains of interest to researchers as evidenced by numerous recent papers. Various upper and lower bounds on ${}_t$ have been discovered. The domination number has, of course, been well studied [8,9].

The average distance of a graph has sometimes been used to provide lower bounds for domination-related invariants, including the domination number itself [4]. One of the first results along these lines is the following theorem due to F. Chung in [1], which originated as a conjecture of the computer program Graffiti [6]. The *independence number* of G, denoted by $\ = \ (G)$, is the cardinality of a largest set of mutually non-adjacent vertices.

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Theorem 1 (Chung)**.** *Let G be a graph. Then*

$$\geq \bar{D},$$

with equality holding if an only if G is complete.

Recently, this theorem has been generalized by Hansen et al. as a result about the *forest number* $f = f(G)$ of a graph G [7]. This is the maximum order of an induced forest of G.

Theorem 2 (Hansen et al.)**.** *Let G be a graph. Then*

$$f \geq 2\bar{D}.$$

This theorem was also motivated by a conjecture of Graffiti [10]. Its proof is based on techniques introduced by Dankelmann in [3]. Dankelmann uses similar techniques in [4] to characterize graphs with fixed order and domination number that have maximum average distance. One can derive the following theorem as a corollary of this characterization (although this is not stated in [4]).

Theorem 3. *Let G be a graph. Then*

$$> \frac{2}{3}\bar{D} - \frac{1}{3}.$$

Moreover, this inequality is best possible.

The proof of Danklelmann's characterization result is lengthy and technical. We give a short direct proof of Theorem 3, as well as the following Theorem 4, which is the main result of our paper. We defer the proofs to a later section.

Theorem 4 (Main Theorem)**.** *Let G be a graph. Then*

$$_t > \bar{D} - \frac{1}{2}.$$

Moreover, this inequality is best possible.

2. Other Definitions

Let $R(k, t, l)$ denote the *binary star* on $k + t + l$ vertices, where the maximal interior path has order t and there are k leaves on one side of the binary star and l leaves on the other. See Figure 1.

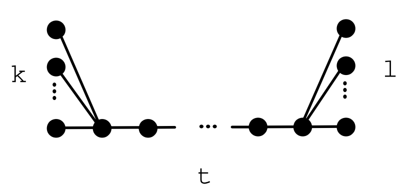

Figure 1. Binary star $R(k, t, l)$.

Now let $R(n, t)$ denote the binary star of order n where the maximal interior path has order t and the leaves are as balanced as possible on each side of the binary star.

A set D of vertices of a graph G is called a *connected dominating set* provided D is a dominating set that induces a connected subgraph of G. The *connected* *domination number* of G, denoted $\,_c = \,_c(G)$, is the cardinality of a smallest connected dominating set in G. A *trunk* for a graph G is a sub-tree (not necessarily induced) that contains the vertices of a dominating set of G. Hence, every spanning tree of G is a trunk for G, and every connected dominating set is the vertex set of some trunk. Standard graph theoretical terms not defined in this paper can be found in [11], for instance.

3. Lemmas

The proof of Lemma 5 involves elementary algebra, counting, and limit arguments; we therefore omit it.

Lemma 5. *For integers $k \geq 0$ and $t \geq 1$,*

$$W(R(k,t,k)) = (t+3)k^2 + (t+2)(t-1)k + \frac{t(t+1)(t-1)}{6}, \textit{and}$$

$$W(R(k,t,k+1)) = (t+3)k^2 + (t+1)^2 k + \frac{t(t+1)(t+2)}{6}.$$

Moreover,

$$W(R(k,t,k)) < W(R(k,t,k+1)) < W(R(k+1,t,k+1)), \textit{and}$$

$$\lim_{k\to\infty} \bar{D}(R(k,t,k)) = \frac{t+3}{2}.$$

The following lemma is proven in [6, Theorem 2].

Lemma 6. *Let G be a graph with a trunk of order $t \geq 1$. Then*

$$\bar{D}(G) \leq \bar{D}(R(n,t)),$$

with equality holding if and only if $G = R(n,t)$.

The next lemma follows by combining the two previous lemmas.

Lemma 7. *Let G be a graph with a trunk of order $t \geq 1$. Then*

$$\bar{D}(G) < \frac{t+3}{2}.$$

An immediate consequence of Lemmas 5 and 7 is the following corollary, which defines the relationship between the minimum order of a connected dominating set of a graph G, denoted $\,_c = \,_c(G)$, and its average distance.

Corollary 8. *Let G be a graph. Then*

$$\,_c > 2\bar{D} - 3.$$

Moreover, this inequality is best possible.

Proof. Let D be a minimum connected dominating set. Then any spanning tree of the subgraph induced by D is a trunk for G. Hence, by Lemma 7,

$$\bar{D}(G) < \frac{\,_c + 3}{2}.$$

To show this inequality is best possible, consider $R(j,t,j)$, where $t \geq 1$ and $j \geq 0$. It is easy to see that $\,_c(R(j,t,j)) = t$. But by Lemma 5,

$$\lim_{j\to\infty} \bar{D}(R(j,t,j)) = \frac{t+3}{2} = \frac{\,_c + 3}{2}.$$

□

One final lemma is needed. The next simple lemma provides some relations that hold for the number of edges induced by dominating sets and their complements. Given a graph G with dominating set D, a vertex $v \notin D$ is *over-dominated* by D if it has two or more neighbors in D. The *over-domination number of* v *with respect to* D, denoted by $O_D(v)$, is one less than the number of neighbors v has in D.

Lemma 9. *Let T be a tree with minimum dominating set D such that the number of components of D is k. Denote the number of edges with both endpoints in D by e_1, the number of edges with both endpoints in $H = T - D$ by e_2, and the number of edges with one endpoint in D and the other endpoint in H by e_3. Moreover, let j be the number of non-trivial components of H with at least two neighbors in D and let l_H be the number of components of H with exactly one neighbor in D (the leaves of H). Then*

a) $e_1 = |D| - k$
b) $e_2 = k - 1 - \sum_{v \in H} O_D(v)$
c) $e_3 = n - |D| + \sum_{v \in H} O_D(v)$
d) $2j + l_H \leq e_3 = k + j + l_H - 1$
e) $n - l_H + 2 + \sum_{v \in H} O_D(v) \leq 2k + |D|$.

Proof. Part a) holds because D induces a forest with k trees. Part c) is true because every vertex in H has a neighbor in D, giving the $n - |D|$, and because the summation contributes the extra edges that have one endpoint D and one in H. Part b) follows immediately from parts a) and c), since $n - 1 = e_1 + e_2 + e_3$ for a tree.

The left hand side of d) comes from the fact that, when counting the edges between D and H, each of the l_H leaves in H contributes exactly one edge while each of the j non-trivial components of H contributes at least two edges. The right hand side of d) follows easily by viewing the components of D together with the components of H as the vertices of a new tree with e_3 edges and $k + j + l_H$ vertices.

From d) we deduce that there are at most $k - 1$ non-trivial components of H, that is, $j \leq k - 1$. Combining this with the right hand side of d) and part c), we arrive at inequality e). □

4. Theorem Proofs

Our strategy for proving Theorem 4 is as follows. Given a minimum total dominating set D of a graph G, we form a particular spanning tree T of G so that D is also a minimum total dominating set of T. Then we apply the lemmas from the previous section to obtain the desired result.

Theorem 4 (Main Theorem) *Let G be a graph. Then*

$${}_t > \bar{D} - \frac{1}{2}.$$

Moreover, this inequality is best possible.

Proof. Let D be a minimum total dominating set of G. Suppose that D has k components. We form a spanning tree T of G such that D is also a minimum total dominating set of T. If G is a tree, then put $T = G$ and we are done. Otherwise, let C be a cycle in G. We delete an edge from C as follows.

i) If C has two consecutive vertices x and y such that $x \notin D$ and $y \notin D$, then delete the edge between them. The set D is still total dominating set for the resulting graph.

ii) Suppose the first case does not apply. If C has two consecutive vertices x and y such that $x \in D$ and $y \notin D$, then delete the edge between them. Since the other neighbor of y on C is necessarily in D (or else the first case applies), the set D is still a total dominating set for the resulting graph.

iii) If neither of the first two cases apply, then all of the vertices of C are in D. Delete any edge of C and the set D is still a total dominating set for the resulting graph.

Repeat this process until all cycles are removed. Call the resulting spanning tree T. Since D is a total dominating set of T, $\ _t(T) \leq |D| = \ _t(G)$. Since the total domination number of a graph is at most the total domination number of any of its spanning trees, $\ _t(G) \leq \ _t(T)$. Thus, $\ _t(T) = |D|$ and D is a minimum total dominating set of T.

Now, let L_H, of cardinality l_H, denote the leaves of T that are in $H = T - D$ (the leaves of T that are not in D). Observe that the sub-tree $T - L_H$ contains the total dominating set D of G and is thereby a trunk for G. From Lemma 7,

$$2\bar{D} - 3 < |T - L_H| = n - l_H.$$

Hence by Lemma 9 part e), and since $2k \leq \ _t$,

$$2\bar{D} - 3 < 2k + \ _t - 2 - \sum_{v \in H} O_D(v) \leq 2\ _t - 2 - \sum_{v \in H} O_D(v) \leq 2\ _t - 2.$$

Rearranging yields the desired inequality.

To show the inequality is best possible, consider $R(j,t,j)$, where $t \equiv 2 \pmod 4$ and $j \geq 0$. It is easy to see that $\ _t(R(j,t,j)) = \frac{t}{2} + 1$. But by Lemma 5,

$$\lim_{j \to \infty} \bar{D}(R(j,t,j)) = \frac{t}{2} + \frac{3}{2} = \ _t + \frac{1}{2}.$$

□

The proof of the theorem provides a necessary condition for $\ _t = \lceil \bar{D} - \frac{1}{2} \rceil$. In the proof we found a spanning tree T of a connected graph G such that a minimum total dominating set of G was also a total dominating set for T. We let $H = T - D$ and found that

$$\ _t > \bar{D} - \frac{1}{2} + \frac{1}{2} \sum_{v \in H} O_D(v).$$

Now if $\ _t = \lceil \bar{D} - \frac{1}{2} \rceil$, then

$$\lceil \bar{D} - \frac{1}{2} \rceil = \ _t \geq \lceil \bar{D} - \frac{1}{2} + \frac{1}{2} \sum_{v \in H} O_D(v) \rceil,$$

which immediately suggests that D may over-dominate at most one vertex of H, and if there is an over-dominated vertex of H, its over-domination number is 1.

To see that there exist graphs in which any spanning tree containing a minimum total dominating set of the graph (as a total dominating set for the spanning tree) over-dominates exacly one vertex (with over-domination number 1) of H and $\ _t = \lceil \bar{D} - \frac{1}{2} \rceil$, consider $R(j,t,j)$, where $t > 1$, $t \equiv 1 \pmod 4$ and $j \geq t$. On the other hand, that this condition is not sufficient for equality is seen in P_{4k+3} (the path on $4k + 3$ vertices) for $k \geq 1$. Any minimum total dominating set D in P_{4k+3} over-dominates exactly one vertex v of $V - D$, and v has over-domination number 1, but $\ _t$ is about one half the number of vertices and $\bar{D}$ is about one third of the number vertices.

Next we present a short and direct proof of Theorem 3. As mentioned previously, this result can be deduced from a result of Dankelmann in [4].

Theorem 3 *Let G be a graph. Then*

$$> \frac{2}{3}\bar{D} - \frac{1}{3}.$$

Moreover, this inequality is best possible.

Proof. Let D be a minimum dominating set of G. Suppose that D has k components. We will form a spanning tree T of G such that D is also a minimum dominating set of T. If G is a tree, then put $T = G$ and we are done. Otherwise, let C be a cycle in G. We delete an edge from C as follows.

i) If C has two consecutive vertices x and y such that $x \notin D$ and $y \notin D$, then delete the edge between them. The set D still dominates the resulting graph.
ii) Suppose the first case does not apply. If C has two consecutive vertices x and y such that $x \in D$ and $y \notin D$, then delete the edge between them. Since the other neighbor of y on C is necessarily in D (or else the first case applies), the set D still dominates the resulting graph.
iii) If neither of the first two cases apply, then all of the vertices of C are in D. Delete any edge of C and the set D still dominates the resulting graph.

Repeat this process until all cycles are removed. Call the resulting spanning tree T. Since D is a dominating set of T, $\ (T) \leq |D| = \ (G)$. Since the domination number of a graph is at most the domination number of any of its spanning trees, $\ (G) \leq \ (T)$. Thus, $\ (T) = |D|$ and D is a minimum dominating set of T.

Now, let L_H, of cardinality l_H, denote the leaves of T that are in $H = T - D$ (the leaves of T that are not in D). Observe that the sub-tree $T - L_H$ contains the dominating set D of G and is thereby a trunk for G. From Lemma 7,

$$2\bar{D} - 3 < |T - L_H| = n - l_H.$$

Hence by Lemma 9 part e), and since $2k \leq 2\ $,

$$2\bar{D} - 3 < 2k + \quad - 2 - \sum_{v \in H} O_D(v) \leq 3 \quad - 2 - \sum_{v \in H} O_D(v) \leq 3 \quad - 2.$$

Rearranging yields the desired inequality.

To show the inequality is best possible, consider the family of stars S_n. Since the average distance in stars can be made arbitrarily close to 2, $\frac{2}{3}\bar{D}(S_n) - \frac{1}{3}$ can be made arbitrarily close to $\ (S_n) = 1$. □

As was the case for total domination number and average distance, one can deduce from the proof a similar necessary condition for equality in $\ = \lceil \frac{2}{3}\bar{D} - \frac{1}{3} \rceil$.

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