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ORIGINAL PAPER

A Characterization of Graphs Where the Independence Number Equals the Radius

Ermelinda DeLaViña · Craig E. Larson · Ryan Pepper · Bill Waller

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Abstract In a classical 1986 paper by Erdös, Saks and Sós every graph of radius r has an induced path of order at least $2r - 1$. This result implies that the independence number of such graphs is at least r. In this paper, we use a result of S. Fajtlowicz about radius-critical graphs to characterize graphs where the independence number is equal to the radius, for all possible values of the radius except 2, 3, and 4. We briefly discuss these remaining cases as well.

Keywords Ciliate · Bipartite number · Forest number · Independence number · Path number · Radius · Scaffold · Tree number

1 Introduction and Key Definitions

We limit our discussion to graphs that are simple, connected, and finite of order n. Although we often identify a graph G with its set of vertices, in cases where we need to be explicit we write $V(G)$. If u, v are a pair of adjacent vertices, we denote the corresponding edge by uv. An independent set is a set of pairwise non-adjacent vertices. We let $\alpha = \alpha(G)$ denote the independence number of G; this is the cardinality of a maximum independent set of G. The eccentricity of a vertex v of G, denoted by $ecc(v)$, is the maximum of the shortest-path distances from v to the other vertices of G. The minimum eccentricity taken over all vertices of G is called the radius of G and is denoted $r = r(G)$. We define the path number of G, denoted by $p = p(G)$, as the

E. DeLaViña · R. Pepper · B. Waller
University of Houston-Downtown, Houston, TX 77002, USA

C. E. Larson (✉)
Virginia Commonwealth University, Richmond, VA 23284, USA
e-mail: clarson@vcu.edu

maximum order of an induced path in the graph. One can make analogous definitions for the tree number, forest number, and bipartite number of G. These invariants are denoted by $t = t(G)$, $f = f(G)$, and $b = b(G)$, respectively. Other more specialized definitions will be introduced immediately prior to their first appearance. Standard graph theoretical terms not defined in this paper can be found in [10].

In a classical 1986 paper by Erdös, Saks and Sós [3], using a proof credited to F. Chung, it is shown that every graph of radius r has an induced path of order at least $2r - 1$. We state this result as Theorem 1.

Theorem 1 [3] *Let G be a graph. Then*

$$p \geq 2r - 1. \tag{1}$$

Some immediate corollaries of Theorem 1 are summarized in the following Theorem 2. The best known of these corollaries is that the independence number of a graph is at least as large as its radius. This result was proven independently at roughly the same time as Theorem 1 by Fajtlowicz and Waller [5], motivated by an early conjecture of the computer program Graffiti [4], as well as by Favaron, Mahéo and Saclé [7]. Neither of the these independent proofs is similar to Chung's proof of the Induced Path Theorem.

Theorem 2 *Let G be a graph. Then each of the following inequalities holds*:

1. $\alpha \geq r$;
2. $t \geq 2r - 1$;
3. $f \geq 2r - 1$; *and*
4. $b \geq 2r$.

The first three inequalities are obvious consequences of Theorem 1. Part 4 was proved by Fajtlowicz [6]. (The r-ciliates introduced in this paper are bipartite and have at least $2r$ vertices).

Although it is easy to find graphs (other than cliques) for which these four inequalities are best possible, the problem of characterizing the case of equality for each lower bound has apparently remained unresolved. Of particular interest has been characterizing those graphs where $\alpha = r$ [2,5–7]. Thus, the main purpose of this paper is to characterize those graphs where $\alpha = r$, when $r \geq 5$. It seems possible our techniques can be modified to characterize the case of equality for the remaining inequalities from Theorem 2 as well, for large enough r. Indeed, because $\alpha \geq b/2 \geq r$, conditions that are sufficient to imply $\alpha = r$ also imply $b = 2r$. Nevertheless, we do not directly address these problems here.

There exist at least two different generalizations of Theorem 1, provided independently by Fajtlowicz [6, Theorem 2], Bacsó and Tuza [1, Theorem 1] and Freed and Melrose [8]. Another of Bacsó and Tuza's theorems (stated here as Theorem 3) moreover characterizes the case of equality for Theorem 1. For an integer k, a vertex v of a graph G is called a *k-center* of G provided each vertex of G is within distance k of v.

Theorem 3 [1,8] *Let G be a graph. Then* $p(G) = 2r(G) - 1$ *if and only if for every connected induced subgraph* H *of* G, $r(H) \leq r(G)$ *and each vertex of* H *is within distance* $r(G) - 2$ *of an* $r(G)$*-center of* H.

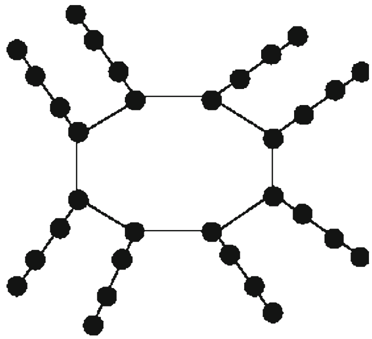


Fig. 1 The 7-ciliate $C(8, 3)$

However, it is the aforementioned 1988 result of Fajtlowicz that plays a key role in this paper. Fajtlowicz proves this result in the context of characterizing *radius-critical* graphs, which are graphs in which every proper induced connected subgraph has radius strictly less than the parent graph. Let $P(n)$ and $C(n)$ denote the path on n vertices and the cycle on n vertices, respectively. Let $C(p, q)$ denote the graph obtained from p disjoint copies of $P(q + 1)$ by linking together one endpoint of each path in a cycle $C(p)$. The graphs $C(2t, r - t)$ have radius r and are referred to as *r-ciliates*. Ciliates include the even paths $P(2r)$ and even cycles $C(2r)$ as the extreme cases $t = 1$ and $t = r$. Figure 1 depicts the 7-ciliate $C(8, 3) = C(2 \cdot 4, 7 - 4)$.

Theorem 4 [6] *Let G be a graph with $r \geq 1$. Then G contains an r-ciliate as an induced subgraph.*

Let G be a graph with $r \geq 1$. Suppose G contains an induced subgraph H such that $H = P(2r)$ or $H = C(2r)$. We call the vertices of H the internal vertices. Enumerate the internal vertices as $h_1, h_2, h_3, \ldots, h_{2r}$, where h_1 is adjacent to h_2, h_2 is adjacent to h_3, and so forth. Let h_i and h_j be two distinct vertices on H. Then we define $\delta(h_i, h_j) = \min\{|j - i|, 2r - |j - i|\}$. Note that if H is a cycle, then $\delta(h_i, h_j)$ is just the shortest-path distance between h_i and h_j with respect to H. If H is a path, let F be the cycle formed from H by joining h_1 and h_{2r}. Then $\delta(h_i, h_j)$ is just the shortest-path distance between h_i and h_j with respect to F. Moreover, we say that h_i and h_j are *consecutive* provided $\delta(h_i, h_j) = 1$. Hence, h_1 and h_{2r} are consecutive.

Next, let $S = V(G) - V(H)$. We call the vertices of S the *external vertices*. Suppose v is an external vertex. Then we let $\delta(v) = \max\{\delta(h_i, h_j) : v \text{ is adjacent to } h_i, h_j\}$. We will later show $\delta(v)$ is well-defined when $\alpha = r$, i.e. each external vertex must be adjacent to at least two internal vertices. We call v a *double vertex* (with respect to H) if v is adjacent to exactly two consecutive internal vertices. Likewise, we call v a *triple vertex* (with respect to H) if v is adjacent to exactly three consecutive internal vertices.

A pair of external vertices is said to be *degenerate* (with respect to H) if the union of their internal neighbors is a subset of three consecutive vertices. A pair of external vertices $\{u, v\}$ is said to be *related* (with respect to H) if the union of their internal neighbors is a set of four consecutive vertices. The edge in H joining the middle two of these four consecutive internal vertices is called the *central edge* associated with the

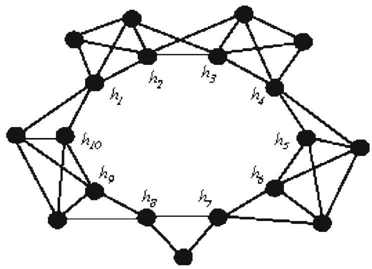


Fig. 2 A 5-scaffold with $H = C(10)$

related pair $\{u, v\}$. On the other hand, a pair of external vertices is said to be *unrelated* (with respect to H) if they are neither degenerate nor related.

Finally, if H is a cycle, properly color the edges of H alternately red and green. If H is a path, imagine the cycle F formed from H by joining h_1 and h_{2r} and properly color the edges of F alternately red and green. In either case, we can assume the edge h_1h_2 is colored red and the (possibly imaginary) edge h_1h_{2r} is colored green. Associate the color red (respectively green) with each pair of related vertices having a central edge that is red (respectively green). Furthermore, we call a double vertex whose neighbors in H or F induce a red edge (respectively, green edge), a *red double* (respectively, *green double*).

A graph G is called an r-*scaffold* provided G has radius r and contains an induced subgraph $H = P(2r)$ or $H = C(2r)$ such that these seven conditions hold.

(1) Every external vertex is either a double or a triple vertex.
(2) Every pair of degenerate vertices is adjacent and no pair of unrelated vertices is adjacent.
(3) Let $\{u, v\}$, $\{x, y\}$ be two pairs of related vertices which are associated with different colors. If the pairs are disjoint with no edges between the first pair and the second pair, or if $v = x$, then either uv is an edge or xy is an edge.
(4) If H is a path, and v is a double vertex whose two internal neighbors are not endpoints of H, then v is red.
(5) If H is a path, and $\{x, y\}$ is a pair of related vertices associated with red whose central edge is not the first or last edge in H, then xy is an edge.
(6) If x is a red double and y is a green double, then they are degenerate. In other words, there are no non-degenerate doubles of opposite color.
(7) If x is a red double (respectively, green double), then all pairs of related vertices that have no common internal neighbors with x and are associated with red (respectively green) are adjacent.

Figure 2 depicts a 5-scaffold with $H = C(10)$. Conditions (1)–(3) are satisfied and none of the other conditions apply.

We now state our main theorem. The proof is outlined in the next two sections.

Theorem 5 (Main Theorem) *Let G be a graph with* $r = 1$ *or* $r \geq 5$. *Then* $\alpha = r$ *if and only if G is an* r-*scaffold.*

In [2], it is show that if G is a graph such that $\alpha = r$, then G has a Hamiltonian path. Several of the results proved in that paper are useful in the next section.

2 Necessity for $\alpha = r$ when $r = 1$ or $r \geq 5$

In this section, we show that if G is a graph with $r = 1$ or $r \geq 5$ such that $\alpha = r$, then G is an r-scaffold. If $\alpha = r = 1$, then G must be a clique, and we set $H = P(2)$. The result is easily seen to be true since all external vertices are doubles and all pairs of external vertices are degenerate. Therefore, we can assume $r \geq 5$. We now prove several lemmas related to G, which taken together imply the desired result.

Lemma 1 [2] *Let G be a graph with $\alpha = r \geq 5$. Then G contains either $P(2r)$ or $C(2r)$ as an induced subgraph. Moreover, if we let H denote an induced $P(2r)$ or $C(2r)$ subgraph, then each vertex of G is either contained in H or is adjacent to H.*

Preferred Ciliate Assumption. Let G be a graph with $\alpha = r \geq 5$. If G contains $C(2r)$ as an induced subgraph, we will henceforth let H denote one of these subgraphs. Otherwise, we will let H denote the induced $P(2r)$ subgraph implied by Lemma 1 that maximizes the number of external vertices y such that $\delta(y) \leq 2$. As before, we put $S = V(G) - V(H)$. It should be emphasized that this result holds for the remainder of this section.

Lemma 2 [9] *Let G be a graph with $\alpha = r \geq 5$. Suppose Q is a non-empty independent set of external vertices and R is the set of internal neighbors of vertices in Q. If $|R| \leq 2|Q|$, then none of the connected components of $H - R$ is a path of odd order.*

Proof By Lemma 1, R is non-empty. Because $H - R$ is a union of disjoint paths, if one of the connected components of the subgraph induced by $H - R$ is a path of odd order, then

$$\alpha(G) \geq \alpha(H - R) + |Q| > \frac{2r - |R|}{2} + |Q| \geq \frac{2r - 2|Q|}{2} + |Q| = r,$$

a contradiction.

Lemma 3 *Let G be a graph with $\alpha = r \geq 5$. If v is an external vertex, then v is adjacent to at least two internal vertices.*

Proof By way of contradiction, suppose h is the unique internal neighbor of v. Now apply Lemma 2 with $Q = \{v\}$ and $R = \{h\}$.

Note that Lemma 3 implies that $\delta(v)$ is well-defined for every external vertex v. The next lemma partially establishes the necessity of Condition (1).

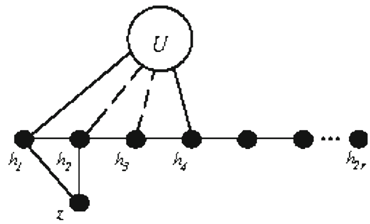
Lemma 4 [2] *Let G be a graph with $\alpha = r \geq 5$. If $H = C(2r)$, then each external vertex is either a double or a triple vertex.*

We should point out that the proof requires $r \geq 5$.

However, establishing the necessity of Condition (1) when $H = P(2r)$ is considerably more difficult, and requires the following sequence of lemmas, culminating in Lemma 11.

Lemma 5 [2] *Let G be a graph with $\alpha = r \geq 5$. If $H = P(2r)$ and v is an external vertex, then $1 \leq \delta(v) \leq 3$. Thus, the internal neighbors of v must be a subset of four consecutive vertices.*

Fig. 3 Referred to in the proof of Lemma 9



Lemma 6 *Let G be a graph with $\alpha = r \geq 5$. Suppose $H = P(2r)$ and v is an external vertex. If $\delta(v) = 3$, the internal neighbors of v cannot be a subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, $\{h_1, h_2, h_{2r-1}, h_{2r}\}$, or $\{h_1, h_2, h_3, h_{2r}\}$.*

This result follows from Lemma 6 in [2].

Lemma 7 *Let G be a graph with $\alpha = r \geq 5$. If $H = P(2r)$, and v is an external vertex, then v cannot be adjacent to both h_1 and h_{2r-1}, or both h_2 and h_{2r}.*

Proof If $H = P(2r)$, then the Preferred Ciliate Assumption implies that G does not contain an induced $C(2r)$. Now by way of contradiction, suppose v is adjacent to both h_1 and h_{2r-1}. Then by Lemma 5, the internal neighbors of v must be a subset of either $\{h_1, h_2, h_{2r-1}, h_{2r}\}$ or $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$. So by Lemma 6, $\delta(v) \leq 2$, which implies the internal neighbors of v are a subset of $\{h_1, h_{2r-1}, h_{2r}\}$. Thus the vertices $\{h_1, h_2, \ldots, h_{2r-1}, v\}$ induce a $C(2r)$ subgraph, which is a contradiction. The case when v is adjacent to both h_2 and h_{2r} is symmetrical.

Lemma 8 *Let G be a graph with $\alpha = r \geq 5$. Suppose $H = P(2r)$ and v is an external vertex such that $\delta(v) = 3$. Then every double vertex whose internal neighbors are the endpoints of H is adjacent to v.*

Proof By way of contradiction, suppose there exists a double vertex z not adjacent to v whose internal neighbors are the endpoints of H. By Lemma 6, the internal neighbors of v are a subset of $\{h_k, h_{k+1}, h_{k+2}, h_{k+3}\}$ for some k, $1 \leq k \leq 2r - 3$. Since $\delta(v) = 3$, v must be adjacent to both h_k and h_{k+3}. Thus the vertices $\{z, h_1, h_2, \ldots, h_k, v, h_{k+3}, \ldots, h_{2r-1}, h_{2r}\}$ induce a $C(2r)$ subgraph, which contradicts the Preferred Ciliate Assumption.

Lemma 9 *Let G be a graph with $\alpha = r \geq 5$. Suppose $H = P(2r)$, and U is a collection of external vertices whose internal neighbors are a subset of $\{h_1, h_2, h_3, h_4\}$. Furthermore, suppose each vertex of U is adjacent to h_4, and U contains a vertex v such that $\delta(v) = 3$. Then there exists a double vertex z such that z is adjacent to h_1 and h_2, and moreover such that z is adjacent to no vertex in U (see Fig. 3).*

Proof Because $\delta(v) = 3$, v is adjacent to h_1 and h_4. Let us consider $ecc(h_{r+1})$. Since $r \geq 5$, all vertices of H are at distance at most $r - 1$ from h_{r+1}, and furthermore h_1, h_2 and h_{2r} are the only vertices in H possibly at distance $r - 1$ from h_{r+1}. Thus, there must exist a vertex z adjacent to at least two of h_1, h_2 and h_{2r}, but no other vertices

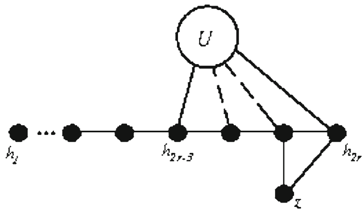


Fig. 4 Referred to in the proof of Lemma 10

in H. If z is adjacent to only h_2 and h_{2r}, this contradicts Lemma 2 with $Q = \{z\}$. Thus, z must be adjacent to h_1. Since z must have two neighbors in H, z must be adjacent to at least one of h_2 or h_{2r}. But z cannot be adjacent to both h_2 and h_{2r}, for this would contradict Lemma 7. Clearly, z is not adjacent to v. Thus, z cannot be adjacent to only h_1 and h_{2r} in H, for this would contradict Lemma 8. Hence, z is adjacent to only h_1 and h_2 in H. Now, assume by way of contradiction, that z is adjacent to some $u \in U$. By definition of the set U, u is adjacent to h_4. But now the distance between z and h_{r+1} is less than r, which contradicts our assumption that the distance from z to h_{r+1} is r.

Lemma 10 *Let G be a graph with $\alpha = r \geq 5$. Suppose $H = P(2r)$ and U is a collection of external vertices whose internal neighbors are a subset of $\{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$. Furthermore, suppose each vertex of U is adjacent to h_{2r-3}, and U contains a vertex v such that $\delta(v) = 3$. Then there exists a double vertex z such that z is adjacent to h_{2r-1} and h_{2r}, and moreover such that z is adjacent to no vertex in U (see Fig. 4).*

Proof The proof is symmetrical to that for Lemma 9.

Lemma 11 *Let G be a graph with $\alpha = r \geq 5$. If $H = P(2r)$, then each external vertex is either a double or a triple vertex.*

Proof Suppose v is an external vertex and let V be the set of internal neighbors of v. By Lemma 5, $1 \leq \delta(v) \leq 3$, and V must be a subset of four consecutive vertices. If $\delta(v) = 1$, then v is adjacent to two consecutive internal vertices. But if $\delta(v) = 2$, and if v is not adjacent to three consecutive internal vertices, this contradicts Lemma 2 with $Q = \{v\}$.

Now suppose $\delta(v) = 3$. By Lemma 6, V cannot be a subset of $\{h_1, h_{2r-2}, h_{2r-1}, h_{2r}\}$, $\{h_1, h_2, h_{2r-1}, h_{2r}\}$, or $\{h_1, h_2, h_3, h_{2r}\}$. This leaves us three cases to consider. Each case will lead to a contradiction, which implies $\delta(v) = 3$.

Case 1. Assume that $V \subseteq \{h_k, h_{k+1}, h_{k+2}, h_{k+3}\}$ where $2 \leq k \leq 2r - 4$. We will show this assumption implies there exists a double vertex z such that z is adjacent to h_1 and h_{2r}. Furthermore, z is not adjacent to v. However, the existence of such a vertex contradicts Lemma 8. Hence, Case 1 cannot occur.

First, let us observe that v must be adjacent to both h_k and h_{k+3}. We consider three subcases.

Subcase 1a. Suppose that $2 \leq k \leq r - 2$. Since v is adjacent to h_{k+3}, it is easily verified that h_{r+1} is at distance at most $r - 1$ from all vertices of H. Moreover, h_1 and

h_{2r} are the only vertices in H possibly at distance $r-1$ from h_{r+1}. Thus, there is a vertex z at distance r from h_{r+1} that is adjacent to h_1 and h_{2r}, and no other vertices of H. Clearly, v is not adjacent to z, otherwise h_{r+1} and z are not at distance r as assumed.

Subcase 1b. If $k = r-1$, then the distance from v to all vertices of H is at most $r-1$. Moreover, h_1 and h_{2r} are the only vertices possibly at distance $r-1$ from v. Thus, there is a vertex z at distance r from v that is adjacent to h_1 and h_{2r}, and clearly v is not adjacent to z.

Subcase 1c. Suppose that $r \leq k \leq 2r-4$. In this case, let us consider $ecc(h_r)$. Vertex h_r is at distance at most $r-1$ from all vertices of H. Moreover, h_1 and h_{2r} are the only vertices possibly at distance $r-1$ from h_r. Thus, there is a vertex z at distance r from h_r that is adjacent to h_1 and h_{2r}, and clearly v is not adjacent to z.

Case 2. Assume that $V \subseteq \{h_1, h_2, h_3, h_4\}$. Let U be the collection external vertices whose internal neighbors are a subset of $\{h_1, h_2, h_3, h_4\}$. Furthermore, suppose each vertex of U is adjacent to h_4. Since $\delta(v) = 3$, we have v is adjacent to both h_1 and h_4. Thus $v \in U$. By Lemma 9, there exists a double vertex z such that z is adjacent to h_1 and h_2, and moreover such that z is adjacent to no vertex in U. Note that the vertices $\{z, h_2, h_3, \ldots, h_{2r}\}$ induce a $P(2r)$ subgraph, which we will denote as H'. We will enumerate the vertices of H' as $\{h'_1 = z, h'_2 = h_2, \ldots, h'_{2r} = h_{2r}\}$. Since v is not adjacent to $h'_1 = z$, we have $\delta(v) \leq 2$ with respect to H'. Now suppose there exists a vertex x external to H' such that $\delta(x) \leq 2$ with respect to H, but $\delta(x) \geq 3$ with respect to H'. Then x is adjacent to z and by Lemma 5, $\delta(x) = 3$ with respect to H'. Applying Lemma 6 to H', we get that x's neighbors in H' must be a subset of $\{h'_1 = z, h'_2 = h_2, h'_3 = h_3, h'_4 = h_4\}$. This implies x is adjacent to $h'_4 = h_4$ and since x cannot be h_1, we have $x \in U$. This contradicts our choice of z, therefore no such vertex x exists. This fact, however, contradicts the Preferred Ciliate Assumption.

Case 3. Assume that $V \subseteq \{h_{2r-3}, h_{2r-2}, h_{2r-1}, h_{2r}\}$. The remainder of this case is symmetrical to Case 2.

The following Lemmas 12–17 show the necessity of the remaining Conditions (2)–(7).

Lemma 12 *Let G be a graph with $\alpha = r \geq 5$. Then every pair of degenerate vertices is adjacent and no pair of unrelated vertices is adjacent.*

Proof Let x and y be a pair of non-adjacent degenerate vertices, and let R be a set of three consecutive vertices of H which contains the union of the internal neighbors of x and y. Put $Q = \{x, y\}$. Because $2|Q| \geq R$, the hypothesis of Lemma 2 is satisfied, but one of the connected components of the subgraph induced by $H - R$ is a path of odd order.

Let x and y be a pair of adjacent unrelated vertices. Suppose first that $H = C(2r)$. Let A be a smallest set of consecutive internal vertices which contains the union of the internal neighbors of x and y. Since x and y are unrelated, $|A| \geq 5$. Suppose first that $|A| \geq 6$. Let a and b be the two endpoints of the path induced by A which are adjacent to x and y, respectively. Now $(H - A) \cup \{a, b, x, y\}$ is a set of vertices which induces a cycle of order no more than $2r - 2 = 2(r-1)$. Consequently, either every vertex of H is at distance less than $r-1$ from x, in which case, since every external vertex of

G is adjacent to a vertex of H, $ecc(x) < r$, a contradiction; or there is a unique vertex of H at distance $r - 1$ from x (when $|A| = 6$), in which case, since every external vertex of G is adjacent to at least two vertices of H, $ecc(x) < r$, a contradiction.

Next we suppose that $|A| = 5$. Again, let a and b be the two endpoints of the path induced by A that are adjacent to x and y, respectively. Now $(H - A) \cup \{a, b, x, y\}$ is a set of vertices that induces a cycle of order $2r - 1$. Since this is an odd cycle, there are exactly two vertices a_1 and a_2 of the cycle at distance $r - 1$ from a and exactly two vertices b_1 and b_2 of the cycle at distance $r - 1$ from b. Moreover, every other vertex of the cycle (induced by $(H - A) \cup \{a, b, x, y\}$) is at distance less than $r - 1$ from a (respectively b). Now, since every vertex not on H is adjacent to at least two vertices of H, this means there is some external vertex a that is adjacent to the two vertices a_1 and a_2 and no other vertices of H, for otherwise, $ecc(a) < r$. Similarly, there is some external vertex b that is adjacent to the two vertices b_1 and b_2 and no other vertices of H. Furthermore, a and b are not adjacent since the distance between b and a is $r - 3$ and the distance between b and b is r. Taking Q to be $\{a , b \}$ and R to be $\{a_1, a_2, b_1, b_2\}$, both connected components of the subgraph induced by $H - R$ are paths of odd order, a contradiction with Lemma 2. Therefore, it cannot be the case that $H = C(2r)$.

Suppose instead that $H = P(2r)$. Let $a = h_r$ and $b = h_{r+1}$ be the bicenter of H. Again let A be a smallest set of consecutive internal vertices which contains the union of the internal neighbors of x and y (recall x and y are a pair of adjacent unrelated vertices). First, assume the internal neighbors of y include both h_1 and h_{2r}. By Lemmas 7 and 11, y must be a double vertex. Because x and y are unrelated, Lemma 5 implies that x must be adjacent to some vertex h_k where $4 \leq k \leq 2r - 3$. Thus $ecc(y) \leq r - 1$, a contradiction. By the same argument, x cannot be adjacent to both h_1 and h_{2r}. Now let u be the vertex of A of smallest index that is adjacent to x, and let v be the vertex of A of largest index that is adjacent to y. We can assume that x and y are labeled so that the index of u is less than the index of any of the neighbors of y on H and the index of v is greater than the index of any of the neighbors of x on H.

Assume first that b is at least as close to h_{2r} as v is with respect to H. Now, if $|A| \geq 6$, then $ecc(b) < r$ (since there is exactly one vertex, namely h_{2r}, at distance $r - 1$ from b and every other vertex of G is adjacent to at least two vertices of H). So, we may assume in this case that $|A| = 5$. Now if $u = h_1$, then the two vertices h_1 and h_{2r} are at distance $r - 1$ from b, and every other vertex of H is at distance less than $r - 1$ from b. Thus there must be an external vertex z adjacent to h_1 and h_{2r} and to no other vertices of H. This means there is an induced $C(2r)$ in G (containing z, h_1, x, y, u, v and h_{2r}), which contradicts the Preferred Ciliate Assumption. On the other hand, if $u = h_1$, then the three vertices, h_1, h_2, and h_{2r} are at distance $r - 1$ from b. In this case, there must be an external vertex z adjacent to either h_1 and h_2 and to no other vertices of H, or to h_1 and h_{2r} and to no other vertices of H. In the first case, the internal neighbors of $\{z, x\}$ are a subset of $\{h_1, h_2, h_3\}$. Thus z and x are degenerate. It follows from the earlier part of this proof that z and x are adjacent, thus $ecc(b) < r$. In the second case, again there is an induced $C(2r)$ in G, which contradicts the Preferred Ciliate Assumption. Consequently, it is not the case that b is at least as close to h_{2r} as v is with respect to H (that is, v is between b and h_{2r} on H).

By a symmetric argument, it follows that u is between a and h_1. First assume $|A| \geq 6$. Now, if x is at least as close to the bicenter $\{a, b\}$ as y is, then $ecc(x) < r$, which contradicts the assumption that the radius is r. Otherwise, $ecc(y) < r$, a contradiction.

So we must assume $|A| = 5$. In this case, we may assume without loss of generality that u is adjacent to a. Now, h_1 and h_{2r} are at distance $r - 1$ from x and all other vertices of H are at distance strictly less than $r - 1$ from x. Thus, there is a vertex z adjacent to both h_1 and h_{2r} and to no other vertices of H. This means that there is an induced $C(2r)$ in G (containing z, h_1, u, x, y, v and h_{2r}), which contradicts the Preferred Ciliate Assumption. Therefore our assumption that x and y are adjacent and unrelated must be false.

Lemma 13 *Let G be a graph with $\alpha = r \geq 5$. Let $\{u, v\}, \{x, y\}$ be two pairs of related vertices which are associated with different colors. If the pairs are disjoint with no edges between the first pair and the second pair, or if $v = x$, then either uv is an edge or xy is an edge.*

Proof Lemmas 4 and 11 imply that each of u, v, x, and y are doubles or triples. Assume $\{u, v\}$ are associated with a red edge and $\{x, y\}$ are associated with a green edge. By way of contradiction, assume neither uv nor xy is an edge. Because we colored h_1h_2 red, the internal neighbors of u and v are $\{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}\}$ for some j, $1 \leq j \leq r$. (When $j = r - 1$, we assume $h_{2j+3} = h_1$. When $j = r$, we assume $h_{2j+1} = h_1, h_{2j+2} = h_2$, and $h_{2j+3} = h_3$.) Likewise, the internal neighbors of x and y are $\{h_{2k+1}, h_{2k+2}, h_{2k+3}, h_{2k+4}\}$ for some k, $0 \leq k \leq r - 1$ (when $k = r - 1$, we assume $h_{2k+3} = h_1$ and $h_{2k+4} = h_2$). Suppose $k \geq j$, and put $Q = \{u, v, x, y\}$. If $|Q| = 4$ then Lemma 2 is violated. If $k = j$, then $|Q| \geq 3$. Let us assume $|Q| = 3$ and $v = x$. Note that in this case u and y must be unrelated, and therefore non-adjacent by Lemma 12. However, the set of internal neighbors of Q is $R = \{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}, h_{2j+4}\}$. This contradicts Lemma 2. Similarly, if $k = j + 1$, then $|Q| = 4$ and the set of internal neighbors of Q is $R = \{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}, h_{2j+4}, h_{2j+5}, h_{2j+6}\}$. This again contradicts Lemma 2. Finally, suppose $k \geq j + 2$. Then $|Q| = 4$ and the set of internal neighbors of Q is $R = \{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}, h_{2k+1}, h_{2k+2}, h_{2k+3}, h_{2k+4}\}$. Thus the set $\{h_{2j+4}, h_{2j+5}, \ldots, h_{2k}\}$ is non-empty and therefore induces a path of odd order, once more contradicting Lemma 2. We can use a symmetrical argument in the case $j > k$.

Lemma 14 *Let G be a graph with $\alpha = r \geq 5$. If $H = P(2r)$, and v is a double vertex whose two internal neighbors are not endpoints of H, then v is red.*

Proof By way of contradiction, suppose v is green. Then v's internal neighbors are h_{2j} and h_{2j+1} for some j, $1 \leq j \leq r - 1$. Now apply Lemma 2 with $Q = \{v\}$ and $R = \{h_{2j}, h_{2j+1}\}$.

Lemma 15 *Let G be a graph with $\alpha = r \geq 5$. If $H = P(2r)$, and $\{x, y\}$ is a pair of related vertices associated with red whose central edge is not the first or last edge in H, then xy is an edge.*

Proof By way of contradiction, suppose x and y are not adjacent. By Lemmas 11 and 14, both x and y must be triples (for otherwise, one must be a green double whose two internal neighbors are not endpoints of H, a contradiction). Moreover, the internal neighbors x and y are $\{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}\}$ for some j, $1 \leq j \leq r-2$. Then the set $\{h_1, h_2, \ldots, h_{2j-1}\}$ is non-empty and therefore induces a path of odd order. This contradicts Lemma 2 with $Q = \{x, y\}$ and $R = \{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}\}$.

Lemma 16 *Let G be a graph with* $\alpha = r \geq 5$. *If* x *is a red double and* y *is a green double, then they are degenerate. Hence, there are no non-degenerate doubles of opposite color.*

Proof By way of contradiction, suppose x and y are not degenerate. Because x and y are doubles of opposite color, then they are unrelated. Hence x and y are not adjacent by Lemma 12. Moreover, the internal neighbors of x are h_{2j+1} and h_{2j+2} for some j, $0 \leq j \leq r-1$. Likewise, the internal neighbors of y are h_{2k} and h_{2k+1} for some k, $1 \leq k \leq r$ (when $k = r$, we assume $h_{2k+1} = h_1$). Because x and y are not related, we have either $2k > 2j+2$ or $2k+1 < 2j+1$. Suppose $2k > 2j+2$. Thus the set $\{h_{2j+3}, h_{2j+4}, \ldots, h_{2k-1}\}$ is non-empty and therefore induces a path of odd order. This contradicts Lemma 2 with $Q = \{x, y\}$ and $R = \{h_{2j+1}, h_{2j+2}, h_{2k}, h_{2k+1}\}$. We can use a symmetrical argument in the case $2k+1 < 2j+1$.

Lemma 17 *Let G be a graph with* $\alpha = r \geq 5$. *If* x *is a red double (respectively, green double), then all pairs of related vertices that have no common internal neighbors with* x *and are associated with red (respectively, green) are adjacent.*

Proof Suppose x is a red double and $\{u, v\}$ is a pair of related vertices associated with red that have no common internal neighbors with x. By way of contradiction, assume u and v are not adjacent. Because x has no common internal neighbors with either u or v, then x is unrelated to either of these vertices. Hence x is not adjacent to either of these vertices by Lemma 12. Moreover, the internal neighbors of x are h_{2j+1} and h_{2j+2} for some j, $0 \leq j \leq r-1$. Likewise, the internal neighbors of u and v are $\{h_{2k}, h_{2k+1}, h_{2k+2}, h_{2k+3}\}$ for some k, $1 \leq k \leq r$. (When $k = r-1$, we assume $h_{2k+3} = h_1$. When $k = r$, we assume $h_{2k+1} = h_1$, $h_{2k+2} = h_2$, and $h_{2k+3} = h_3$.) Because x is unrelated to either u or v, we have $|k-j| \geq 2$. Suppose $k > j$. Thus the set $\{h_{2j+3}, h_{2j+4}, \ldots, h_{2k-1}\}$ is non-empty and therefore induces a path of odd order. This contradicts Lemma 2 with $Q = \{x, u, v\}$ and $R = \{h_{2j+1}, h_{2j+2}, h_{2k}, h_{2k+1}, h_{2k+2}, h_{2k+3}\}$. We can use symmetrical arguments in the cases $j > k$ or x is green.

3 Sufficiency for $\alpha = r$ when $r = 1$ or $r \geq 5$

In this section, we show that if G is an r-scaffold with $r = 1$ or $r \geq 5$, then $\alpha = r$. If $r = 1$, then G must be a clique, and so we are finished. Therefore, we can assume $r \geq 5$. Because G is an r-scaffold, G contains an induced subgraph $H = P(2r)$ or $H = C(2r)$ such that Conditions (1)–(7) are satisfied with respect to H.

We consider two major cases, and several subcases. First, though, let R_{2j+1} be the set of external vertices adjacent to h_{2j+2} whose internal neighbors are a subset of $\{h_{2j}, h_{2j+1}, h_{2j+2}, h_{2j+3}\}$ for every $j, 0 \le j \le r-1$ (When $j = 0$, we assume $h_{2j} = h_{2r}$, and when $j = r-1$, we assume $h_{2j+3} = h_1$). Note that $h_{2j+1}h_{2j+2}$ is a red edge. Likewise, let G_{2k} be the set of external vertices adjacent to h_{2k+1} whose internal neighbors are a subset of $\{h_{2k-1}, h_{2k}, h_{2k+1}, h_{2k+2}\}$ for every $k, 1 \le k \le r$ (when $k = r$, we assume $h_{2k+1} = h_1$ and $h_{2k+2} = h_2$). Note that $h_{2k}h_{2k+1}$ is a green edge (but this edge is imaginary when $k = r$ and $H = P(2r)$).

Then if x is a red double vertex, x is an element of G_{2j} and R_{2j+1} for some $j, 0 \le j \le r-1$ (when $j = 0$, we assume $G_{2j} = G_{2r}$). Moreover, if x is a green double, then x is an element of R_{2k-1} and G_{2k} for some k, $1 \le k \le r$. Likewise, if x is a triple, either x is an element of G_{2j} and R_{2j+1} for some $j, 0 \le j \le r-1$ (when $j = 0$, we assume $G_{2j} = G_{2r}$); or, x is an element of R_{2k-1} and G_{2k} for some k, $1 \le k \le r$. Thus by Condition (1), each external vertex is an element of R_p for some $p, 1 \le p \le 2r-1$, and also an element of G_q for some $q, 2 \le q \le 2r$.

Let I be a maximum independent set in G. In light of Conditions (1) and (2), it is easily seen that $|I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})| \le 2$ for every $j, 0 \le j \le r-1$. Likewise, $|I \cap (G_{2k} \cup \{h_{2k}, h_{2k+1}\})| \le 2$ for every $k, 1 \le k \le r$.

Case 1. $H = C(2r)$.

Proof We consider three subcases.

Subcase 1a. Suppose for every $j, 0 \le j \le r-1$, that $I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ does not contain two related vertices associated with the central edge $h_{2j+1}h_{2j+2}$. Moreover suppose there does not exist any green double vertices. We will show $|I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})| \le 1$ for every possible value of j. By way of contradiction, suppose $u, v \in I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ for some $j, 0 \le j \le r-1$. If both u and v are doubles, then they must both be red and adjacent to h_{2j+1} and h_{2j+2}. Therefore they are degenerate, and by Condition (2), they are adjacent. If u is a triple and v is a (red) double, or vice versa, then again they must be degenerate, and therefore adjacent. Furthermore, both are adjacent to h_{2j+1} and h_{2j+2}. Finally, if both u and v are triples, both are adjacent to h_{2j+1} and h_{2j+2}, and they are either degenerate or related with central edge $h_{2j+1}h_{2j+2}$. If the former is true, then they are adjacent by Condition (2). The latter cannot be true as u and v would be related vertices, contradicting our assumption. Clearly either $h_{2j+1} \notin I$ or $h_{2j+2} \notin I$. So in any event, by Condition (1), $|I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})| \le 1$ for every possible value of j.

Note it follows from our observation prior to Case 1 that $V(G) = \cup_{j=0}^{r-1}(R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$. Then,

$$\begin{aligned}
\alpha &= |I| \\
&= |I \cap (\cup_{j=0}^{r-1}(R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&= |\cup_{j=0}^{r-1}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&\le \sum_{j=0}^{r-1} |I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})| \\
&\le r.
\end{aligned}$$

But $\alpha \geq r$ by Theorem 2, so we have $\alpha = r$.

Subcase 1b. Suppose for every j, $0 \leq j \leq r-1$, that $I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ does not contain two related vertices associated with the central edge $h_{2j+1}h_{2j+2}$. But now suppose there exists a green double v. Thus v is an element of R_{2k-1} and G_{2k} for some k, $1 \leq k \leq r$. If $|I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})| \leq 1$ for all possible values of j, then we can show $\alpha = r$ as in Subcase 1a. Therefore, assume otherwise. By our suppositions, without loss of generality in the argument that follows, we can additionally assume that $k = 1$. Note that v is necessarily adjacent to h_2 and h_3. Since any pair of vertices in R_1 must be adjacent, if $|I \cap (R_1 \cup \{h_1, h_2\})| = 2$, then $h_1 \in I \cap (R_1 \cup \{h_1, h_2\})$. Assume $I \cap (R_1 \cup \{h_1, h_2\}) = \{h_1, v\}$.

Claim There exists an integer m, $3 \leq m \leq 2r-1$, such that $|I \cap (R_m \cup \{h_m, h_{m+1}\})| = 0$.

Proof of Claim By way of contradiction, suppose $|I \cap (R_j \cup \{h_j, h_{j+1}\})| \geq 1$ for $j = 3, 5, \ldots, 2r-1$. We show this implies none of the internal vertices $h_1, h_3, h_5, \ldots, h_{2r-1}$ is contained in I, contradicting the assumption $I \cap (R_1 \cup \{h_1, h_2\}) = \{h_1, v\}$. First, clearly $h_3 \notin I$ since h_3 is adjacent to v. Because $|I \cap (R_3 \cup \{h_3, h_4\})| \geq 1$, then necessarily $|I \cap (R_3 \cup \{h_3, h_4\})| = 1$ and obviously there must exist a vertex $x = h_3$ such that $x \in I \cap (R_3 \cup \{h_3, h_4\})$. By Condition (2), x cannot be a red double, because x would be degenerate to v. Thus x is a triple, x is a green double (adjacent to h_4 and h_5), or $x = h_4$. But if x is a triple adjacent to h_2, h_3, and h_4, then x and v are degenerate, again a contradiction by Condition (2). Note that any of the remaining possibilities for x imply $h_5 \notin I$. Put $z_3 = x$.

Now, consider $I \cap (R_5 \cup \{h_5, h_6\})$. But if $h_5 \notin I$ and $|I \cap (R_5 \cup \{h_5, h_6\})| \geq 1$, then necessarily $|I \cap (R_5 \cup \{h_5, h_6\})| = 1$ and obviously there must exist a vertex $x = h_5$ such that $x \in I \cap (R_5 \cup \{h_5, h_6\})$. By Condition (6), x cannot be a red double, because x is not degenerate to v. Thus x is a triple, x is a green double (adjacent to h_6 and h_7), or $x = h_6$. Suppose x is a triple adjacent to h_4, h_5, and h_6. Then $z_3 = h_4$, since $z_3 \in I$. Hence z_3 is a green double (adjacent to h_4 and h_5), or z_3 is a triple adjacent to h_3, h_4, and h_5. In the former case, z_3 and x are degenerate, a contradiction. In the latter case, $\{v, z_3\}$ is a pair of related vertices in $I \cap (R_3 \cup \{h_3, h_4\})$ associated with edge h_3h_4, contradicting the assumption that no such pair exists. Hence x is a green double, x is a triple adjacent to h_5, h_6, and h_7, or $x = h_6$. In any event, x is adjacent to h_7, which implies $h_7 \notin I$. Put $z_5 = x$.

Next, consider $I \cap (R_7 \cup \{h_7, h_8\})$. But if $h_7 \notin I$ and $|I \cap (R_7 \cup \{h_7, h_8\})| \geq 1$, then necessarily $|I \cap (R_7 \cup \{h_7, h_8\})| = 1$ and obviously there must exist a vertex $x = h_7$ such that $x \in I \cap (R_7 \cup \{h_7, h_8\})$. By Condition (6), x cannot be a red double, because x is not degenerate to v. Thus x is a triple, x is a green double (adjacent to h_8 and h_9), or $x = h_8$. Suppose x is a triple adjacent to h_6, h_7, and h_8. Then $z_5 = h_6$, since $z_5 \in I$. Hence z_5 is a green double (adjacent to h_6 and h_7), or z_5 is a triple adjacent to h_5, h_6, and h_7. In the former case, z_5 and x are degenerate, a contradiction. In the latter case, $\{z_5, x\}$ is a pair of related vertices in $I \cap (R_5 \cup \{h_5, h_6\})$ associated with edge h_5h_6, contradicting the assumption that no such pair exists. Hence x is a green double, x is a triple adjacent to h_7, h_8, and h_9, or $x = h_8$. Put $z_7 = x$.

Continuing in the same manner, it follows that $h_9 \notin I, h_{11} \notin I, \ldots, h_{2r-1} \notin I$, and $h_1 \notin I$. But we showed that $h_1 \in I$. This contradiction completes the proof of the claim.

Put $k_1 = 1$. Using the claim, let $f(k_1)$ be the smallest integer such that $3 \leq f(k_1) \leq 2r-1$ and $|I \cap (R_{f(k_1)} \cup \{h_{f(k_1)}, h_{f(k_1)+1}\})| = 0$. From the proof of the claim we can assume $|I \cap (R_j \cup \{h_j, h_{j+1}\})| = 1$ for $j = 3, 5, \ldots, f(k_1) - 2$. Let $k_2 > k_1$ be the smallest integer such that $|I \cap (R_{k_2} \cup \{h_{k_2}, h_{k_2+1}\})| = 2$. If no such integer exists, then we quit. Otherwise $k_2 \geq f(k_1) + 2$, and as in the claim we can argue there exists an integer m, $k_2+2 \leq m \leq 2r-1$, such that $|I \cap (R_m \cup \{h_m, h_{m+1}\})| = 0$. Assume $f(k_2)$ is the smallest such integer m. Moreover, we can assume $|I \cap (R_j \cup \{h_j, h_{j+1}\})| = 1$ for $j = k_2 + 2, \ldots, f(k_2) - 2$. We continue in this manner until we must quit, say after k_p is defined, $p \geq 1$.

Let $K = \{k_1, k_2, \ldots, k_p\}$ and $F = \{f(k_1), f(k_2), \ldots, f(k_p)\}$. Then $K \cap F = \emptyset$. Note it follows from our observation prior to Case 1 that $V(G) = \cup_{j=0}^{r-1}(R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$. Therefore,

$$\begin{aligned}
\alpha &= |I| \\
&= |I \cap (\cup_{j=0}^{r-1}(R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&= |\cup_{j=0}^{r-1}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&= |(\cup_{j\in K}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))) \\
&\quad \cup(\cup_{j\in F}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))) \\
&\quad \cup(\cup_{j\notin(K\cup F)}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})))| \\
&\leq |\cup_{j\in K}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&\quad +|\cup_{j\in F}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&\quad +|\cup_{j\notin(K\cup F)}(I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\}))| \\
&\leq 2p + 0p + 1(r - 2p) = r.
\end{aligned}$$

As before, we have $\alpha = r$ by Theorem 2.

Subcase 1c. Suppose for some j, $0 \leq j \leq r-1$, that $I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ contains two related vertices u and v associated with the central (red) edge $h_{2j+1}h_{2j+2}$. Moreover suppose for every k, $1 \leq k \leq r$, that $I \cap (G_{2k} \cup \{h_{2k}, h_{2k+1}\})$ does not contain two related vertices associated with the central (green) edge $h_{2k}h_{2k+1}$. If there does not exist any red doubles, using an argument symmetrical to Subcase 1a above, we can show that $\alpha = r$. If there does exist a red double, using an argument symmetrical to Subcase 1b above, we can show again that $\alpha = r$. Therefore assume for some k, $1 \leq k \leq r$, that $I \cap (G_{2k} \cup \{h_{2k}, h_{2k+1}\})$ contains two related vertices x and y associated with the central (green) edge $h_{2k}h_{2k+1}$. By Condition (3), there must be an edge between $\{u, v\}$ and $\{x, y\}$, a contradiction.

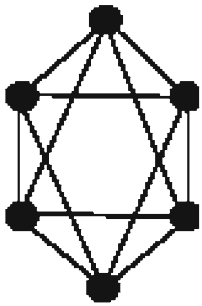
This completes the proof of Case 1.

Case 2. $H = P(2r)$.

Proof We consider three more subcases.

Subcase 2a. Suppose for every j, $0 \leq j \leq r-1$, that $I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ does not contain two related vertices associated with the central edge $h_{2j+1}h_{2j+2}$. Moreover suppose there does not exist any green double vertices. We can now argue as in Subcase 1a to show that $\alpha = r$.

Fig. 5 A graph with $\alpha = r = 2$ but not a 2-scaffold



Subcase 2b. Suppose for every j, $0 \le j \le r-1$, that $I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ does not contain two related vertices associated with the central edge $h_{2j+1}h_{2j+2}$. But now suppose there exists a green double v. By Condition (4), v is adjacent to h_1 and h_{2r}. Thus v is an element of R_{2r-1} and G_{2r}. If $|I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})| \le 1$ for all possible values of j, then we can show $\alpha = r$ as in Subcase 1a. Therefore, assume otherwise. By our suppositions, without loss of generality, we can additionally assume that $I \cap (R_{2r-1} \cup \{h_{2r-1}, h_{2r}\}) = \{h_{2r-1}, v\}$. We can now argue as in Subcase 1b to show that $\alpha = r$.

Subcase 2c. Suppose for some j, $0 \le j \le r-1$, that $I \cap (R_{2j+1} \cup \{h_{2j+1}, h_{2j+2}\})$ contains two related vertices u and v associated with the central (red) edge $h_{2j+1}h_{2j+2}$. By Condition (5), $j = 0$ or $j = r-1$. Thus either $h_1 \notin I$ or $h_{2r} \notin I$. Moreover suppose for every k, $1 \le k \le r$, that $I \cap (G_{2k} \cup \{h_{2k}, h_{2k+1}\})$ does not contain two related vertices associated with the central (green) edge $h_{2k}h_{2k+1}$. If there does not exist any red doubles, then we can use an argument symmetrical to Subcase 1a to show that $\alpha = r$. On the other hand, if there does exist a red double, we can use an argument symmetrical to Subcase 1b to show that $\alpha = r$.

Therefore assume for some k, $1 \le k \le r$, that $I \cap (G_{2k} \cup \{h_{2k}, h_{2k+1}\})$ contains two related vertices x and y associated with the central (green) edge $h_{2k}h_{2k+1}$. By Condition (3), there must be an edge between $\{u, v\}$ and $\{x, y\}$, a contradiction.

This completes the proof of Case 2, and also of Theorem 5.

4 The Remaining Cases $r = 2, 3$, or 4

The graph shown in Fig. 5 has $\alpha = r = 2$, but is not a 2-scaffold (it does not contain an induced path on $2r = 4$ vertices and, for any induced cycle H on $2r = 4$ vertices, the external vertices are not doubles or triples, violating Condition (1) of an r-scaffold). Hence the Main Theorem is not valid in this case.

On the other hand, we conjecture the Main Theorem remains valid for $\alpha = r = 3$ and $\alpha = r = 4$.

Conjecture 1 *Let G be a graph with $r = 1$ or $r \ge 3$. Then $\alpha = r$ if and only if G is an r-scaffold.*

However, our proof of the Main Theorem does not extend to Conjecture 1, as demonstrated by the following two graphs. The graph shown in Fig. 6 has $\alpha = r = 3$ and

Fig. 6 A graph with $\alpha = r = 3$, and a 3-scaffold, but not with respect to the indicated induced $C(6)$

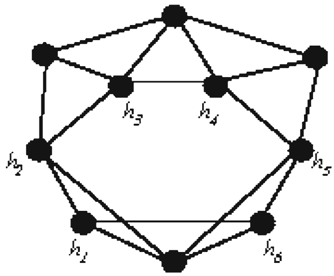


Fig. 7 A graph with $\alpha = r = 4$, and a 4-scaffold, but not with respect to the indicated induced $C(8)$

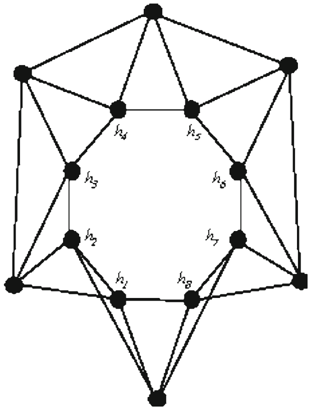
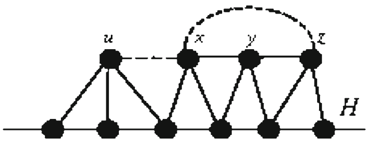


Fig. 8 Referred to in Condition (8)



is a 3-scaffold, but not with respect to indicated induced $C(6)$. The graph shown in Fig. 7 has $\alpha = r = 4$ and is a 4-scaffold, but not with respect to the indicated induced $C(8)$. In both graphs, this is because the bottom-most external vertex is adjacent to four internal vertices. Thus these graphs violate Lemma 4 in the proof.

The definition of an r-scaffold requires a priori that the scaffold has a given radius r. We conjecture that we can weaken the definition of scaffolds, and therefore strengthen the statement of the Main Theorem, as follows. Let $k \geq 1$ be an integer. A graph G will be called a *k-scaffold* provided G contains an induced subgraph $H = P(2k)$ or $H = C(2k)$ satisfying Conditions (1)–(7) listed previously, and moreover such that:

(8) Let x, y and z be distinct double vertices such that x and y are degenerate with only one common internal neighbor, and y and z are degenerate with only one common internal neighbor. Assume x and z are not degenerate. If u is a triple vertex related to x not degenerate with y, then ux and xz cannot both be edges (see Fig. 8).

Fig. 9 Referred to in Condition (9)

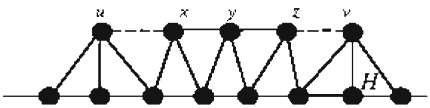


Fig. 10 Referred to in Condition (10)

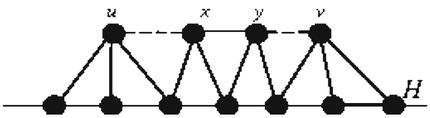
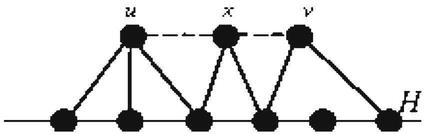


Fig. 11 Referred to in Condition (11)



(9) Let x, y and z be distinct double vertices such that x and y are degenerate with only one common internal neighbor, and y and z are degenerate with only one common internal neighbor. Assume x and z are not degenerate. If u is a triple vertex related to x not degenerate with y, and v is a triple vertex related to z not degenerate with y, then ux and zv cannot both be edges (see Fig. 9).

(10) Let x and y be double vertices such that x and y are degenerate with only one common internal neighbor. If u is a triple vertex related to x not degenerate with y, and v is a triple vertex related to y not degenerate with x, then ux and yv cannot both be edges (see Fig. 10).

(11) Let x be a double vertex related to two triple vertices u, v such that u and v have no common internal neighbors. Then ux and xv cannot both be edges (see Fig. 11).

Conjecture 2 *Let G be a graph and let $k = 1$ or $k \geq 5$. Then $\alpha(G) = r(G) = k$ if and only if G is a k-scaffold.*

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